A new sound mode in liquid ^{4}He ?

R. Schaefer and T. Fliessbach
University of Siegen, Fachbereich Physik
D-57068 Siegen, Germany

Abstract

This letter is based on the hypothesis of a small entropy content of the superfluid fraction of liquid helium. We show that such a superfluid entropy gives rise to a new sound mode in a ring-shaped superleak. This mode is named sixth sound. We propose an experiment by which its sound velocity and thereby the superfluid entropy can be measured. A negative experiment would yield a new upper limit for the superfluid entropy.

PACS numbers: 67.40.Pm, 67.40.Bz, 67.40.Kh

Published in Physics Letters **A 187** (1994) 88

Many experiments show that the entropy S_{s} of the superfluid fraction in He II is small if not zero. The experimental determination of S_{s} (or of an upper limit) has been undertaken by many researchers. The last experiment (to our knowledge) which is explicitly devoted to this question has been performed by Glick and Werntz [1]. The result was that the ratio S_{s}/S is less than 3%; earlier experiments cited in Ref. [1] are less accurate. Singsaas and Ahlers [2] have interpreted their second sound measurement as that of entropy and found no difference to the true (caloric) entropy. The main uncertainty comes here from that in the absolute values of the caloric entropy (about 1 to 2%). Summarizing one may say that a ratio S_{s}/S of the order of one percent is not excluded by the experiment.

In nearly all theoretical approaches S_{s} is taken to be zero. As discussed by Putterman [3] this is an *assumption*. Microscopically, $S_{\mathrm{s}} = 0$ follows from the identification of the superfluid density ρ_{s} with the square of a macroscopic wave function [4]. This identification is plausible but nevertheless an assumption. An early investigation of the effects of a non-vanishing superfluid entropy is given in Ref. [5].

The present letter is based on the hypothesis of a small, non-vanishing superfluid entropy S_{s}. The main result is a proposal for an experiment by which a small superfluid entropy could be measured. In case of a negative result the experiment would yield a considerably lower upper limit for S_{s}/S.

We present a plausibility argument for a small, non-vanishing superfluid entropy; the results of this letter are, however, independent of this argument. We refer to the well-known suggestion [6, 7, 3] of a close connection between the Bose-Einstein-condensation of the ideal Bose gas (IBG) and the λ-transition of liquid ^{4}He. We contrast this with the following discrepancy between liquid helium and the IBG. The critical behaviour of the condensate fraction of the IBG is

$$\frac{\rho_0}{\rho} \propto |t|^{2\beta}, \qquad \beta = \frac{1}{2}\,, \tag{1}$$

where $t = (T - T_\lambda)/T_\lambda$ is the relative temperature. In contrast to (1) the superfluid fraction of He II behaves roughly like

$$\frac{\rho_{\mathrm{s}}}{\rho} \propto |t|^{2\nu}, \qquad \nu \approx \frac{1}{3}\,. \tag{2}$$

The suggested connection between the Bose-Einstein-condensation and the λ-transition is in conflict with $\beta > \nu$ (implying $\rho_0 \ll \rho_{\mathrm{s}}$ just below the transition). This conflict can be resolved by assuming that *non-condensed particles move coherently with the condensate* [8]. Such a coherent motion can be described by multiplying the real single particle functions of non-condensed particles by the complex phase factor of the condensate. The superfluid density ρ_{s} is then made up by the condensate density ρ_0 plus low momentum

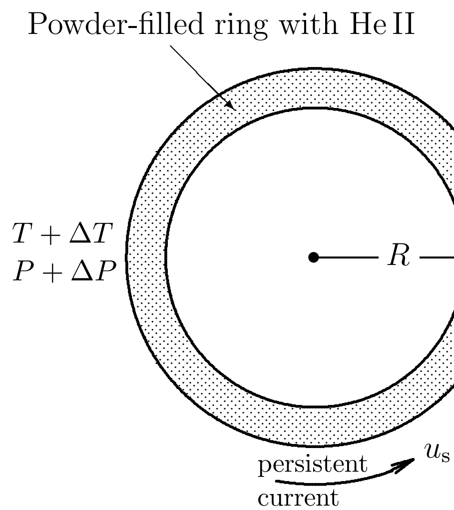

Figure 1: A circular persistent current flows through a ring-shaped superleak. In the common two-fluid model the current may be accompanied by a static fountain pressure (FP) gradient $\Delta P/\Delta T$. If the superfluid entropy is non-zero such a FP gradient is no longer static. It becomes an oscillating mode.

non-condensed particles; as a whole, $\rho_{\rm s}$ can no longer be identified with the square of a macroscopic wave function $|\psi|^2$. The contributing non-condensed particles carry entropy. This leads to a non-vanishing superfluid entropy $S_{\rm s}$.

Fitting '$\rho_{\rm s} = \rho_0 +$ non-condensed particles' to the known superfluid density yields an estimate of the superfluid entropy [8]. This model for $S_{\rm s}$ implies $S_{\rm s} \to 0$ for $T \to T_\lambda$ (because of $\rho_{\rm s} \to 0$) and for $T \to 0$ (because of $\rho_0/\rho \to 1$). Ref. [8] predicts a maximum value of $S_{\rm s}/S$ of about 1% at $T \approx 2\,$K.

The remainder of this letter is independent of the arguments presented in favour of a non-vanishing entropy. The investigation is based on the hypothesis of a small non-vanishing superfluid entropy $S_{\rm s}$.

Generally one expects corrections of the order $S_{\rm s}/S$ for various quantities in He II like the fountain pressure or sound velocities. An investigation [9] of standard experiments shows that these corrections are too small to be readily detected (for $S_{\rm s}/S \sim 1\%$). A non-vanishing superfluid entropy leads, however, to a new sound mode in a ring-shaped superleak [9] as will be shown in the following. This new mode is named sixth sound. Based on the sixth sound, we propose an experiment by which a possible, small superfluid entropy can be measured.

We consider the following two experiments in a ring-shaped superleak (Fig.1):

(i) A persistent current flows with constant velocity $u_{\rm s}$ along the ring. The temperature T and pressure P are constant.

(ii) Along the ring there is a static temperature and pressure gradient. The ratio $\Delta P/\Delta T$ is that of the fountain pressure (FP). The superfluid velocity $u_{\rm s}$ vanishes.

Each of these (meta-) stable configurations may be established independently of whether the superfluid entropy vanishes or not. We consider now the combination of (i) and (ii), that is a persistent current together with a FP gradient:

$S_{\rm s} = 0$: The persistent current carries no entropy. It is compatible with a static FP gradient of T and P; the static entropy balance is unaffected.

$S_{\rm s} \neq 0$: The persistent current carries entropy. For non-constant temperature and pressure this implies a net entropy current and consequently a time-dependence in the entropy continuity equation. *In the presence of a persistent current the FP gradient is non-static.*

For $S_{\rm s} \neq 0$ the non-static combination of (i) and (ii) leads to an oscillation of the FP amplitudes ΔP and ΔT. The discussed static limits imply that the frequency of this oscillation approaches zero for $u_{\rm s} \to 0$ or $S_{\rm s} \to 0$.

The derivation of the new sound mode (or FP oscillation) is based on the equations of motion of the two-fluid model [3] which are adequately modified for $S_{\rm s} \neq 0$. The normal phase is clamped in the superleak,

$$\mathbf{u}_{\rm n} = 0 \, . \tag{3}$$

This condition makes the continuity equation for the momentum (Euler equation) obsolete. The continuity equation for the mass and the entropy, and the equation for the superfluid motion are

$$\begin{aligned} \partial_t \rho + \boldsymbol{\nabla}(\rho_{\rm s} \mathbf{u}_{\rm s}) &= 0 \, , &(4)\\ \partial_t (\rho s) + \boldsymbol{\nabla}(\rho_{\rm s} s_{\rm s} \mathbf{u}_{\rm s}) &= 0 \, , &(5)\\ m \, \partial_t \mathbf{u}_{\rm s} + m (\mathbf{u}_{\rm s} \boldsymbol{\nabla}) \mathbf{u}_{\rm s} &= -\boldsymbol{\nabla}(\mu - \mu_{\rm s}) \, . &(6) \end{aligned}$$

Here $s = S/N$ and $s_{\rm s} = S_{\rm s}/N_{\rm s}$ denote the entropy per particle, $\rho = mN/V$ the mass density of the atoms (mass m), $\mathbf{u}$ the velocity field, μ the chemical potential and $\partial_t = \partial/\partial t$. The indices n and s refer to the normal and superfluid component, respectively. For a simplified discussion all dissipative terms are omitted.

Eq. (4) is not changed by $s_{\rm s} \neq 0$. The modification in (5) is obvious. The correction term $\boldsymbol{\nabla}\mu_{\rm s}$ in (6) needs, however, some explanation. Because of $\operatorname{curl} \mathbf{u}_{\rm s} = 0$ the r.h.s. of (6) must be a gradient field. The FP is determined by the static limit of (6). In the common two-fluid model the well-known

$(dP/dT)_{\rm FP} = s/v$ follows from $d\mu = -s\,dT + v\,dP = 0$ where $v = V/N$. In $(dP/dT)_{\rm FP} = s/v$ we replace s by $s - s_{\rm s}$ because the FP is due to the missing entropy of the superfluid component. This implies

$$\frac{\partial \mu_{\rm s}(T,P)}{\partial T} = -s_{\rm s}(T,P)\,. \tag{7}$$

Together with $s_{\rm s}$ the discussed modification of the two-fluid equations should vanish at T_λ. Eq. (7) and $\mu_{\rm s}(T_\lambda, P) = 0$ define the correction term $\mu_{\rm s}$. This small term will not be used explicitly in the following.

A detailed investigation of the modified two-fluid model (including the $\mu_{\rm s}$-term) and of its sound solutions is given in Ref. [9] and will be published elsewhere. This letter is restricted to a simplified derivation of the sixth sound velocity. For this purpose we consider the ring experiment of Figure 1. The position (middle line) of the ring may be described by

$$\mathbf{r}_{\rm ring} = (R\cos\phi,\; R\sin\phi,\; 0), \qquad \phi = 0\ldots 2\pi\,. \tag{8}$$

The thickness of the ring is assumed to be small compared to the radius R. Then all $\mathbf{r}$-dependences reduce to ϕ-dependences, and the supervelocity is parallel to the ring, $\mathbf{u}_{\rm s}(\mathbf{r},t) = u_{\rm s}(\phi,t)\,\mathbf{e}_\phi$. As independent variables we choose the temperature T, the pressure P and the velocity $u_{\rm s}$. Using

$$\begin{aligned} T(\phi,t) &= T_0 + \Delta T\,\exp({\rm i}\,[\,kR\phi - \omega t]\,)\,, &(9)\\ P(\phi,t) &= P_0 + \Delta P\,\exp({\rm i}\,[\,kR\phi - \omega t]\,)\,, &(10)\\ u_{\rm s}(\phi,t) &= u_{\rm s,0} + \Delta u_{\rm s}\,\exp({\rm i}\,[\,kR\phi - \omega t]\,) &(11)\end{aligned}$$

the equations (4) – (6) are linearized in the amplitudes ΔT, ΔP and $\Delta u_{\rm s}$. For the considered geometry we have three equations for the variables T, P and $u_{\rm s}$ yielding three sound velocities $c = \omega/k$: Two solutions ($\pm c_4$) describe the fourth sound; the third solution (c_6) is a new sound mode which we call *sixth sound*. In the limit $s_{\rm s} = 0$ we obtain $c_6 = 0$ and the sixth sound reduces to a static FP gradient. (The number of equations and solutions is not altered by $s_{\rm s} \neq 0$ because $s_{\rm s}$ is not a new independent variable.)

We derive now the sixth sound solution in a simplified way. Neglecting the l.h.s., eq. (6) can be solved by $\mu - \mu_{\rm s} = $ const. or by

$$\Delta P \approx \frac{s}{v}\,\Delta T\,. \tag{12}$$

The $\mu_{\rm s}$-term in (6) implies a correction of the relative size $s_{\rm s}/s$ which is neglected in (12). The l.h.s. of (6) is not taken into account because it is very small compared to the major terms on the r.h.s.; for example $|m\,\partial_t u_s/s\,\nabla T| \sim |m\,\omega\,\Delta u_s/k\,s\,\Delta T| \sim |m\,\omega\,u_s/k\,s\,T| \ll m\,u_{\rm s}^2/s\,T \ll 1$.

The considered mode is not a compression mode. Therefore, we set $\rho \approx$ const. and solve the continuity equation (4) by

$$\rho \approx \text{const.} \quad \text{and} \quad \rho_{\rm s}\, u_{\rm s} \approx \text{const.}\,. \tag{13}$$

Thermodynamic quantities may depend on T, P and ${u_{\rm s}}^2$. We omit the dependence on the small quantity ${u_{\rm s}}^2$, in particular $s = s(T,P)$ and $s_{\rm s} = s_{\rm s}(T,P)$. We evaluate Δs and $\Delta s_{\rm s}$ for the FP amplitudes (12):

$$\Delta s \;=\; \left(\frac{\partial s}{\partial T}\right)_{P} \Delta T + \left(\frac{\partial s}{\partial P}\right)_{T} \Delta P \approx \left(\frac{\partial s}{\partial T}\right)_{\mu} \Delta T = \frac{c_\mu}{T}\, \Delta T\,, \tag{14}$$

$$\Delta s_{\rm s} \;=\; \left(\frac{\partial s_{\rm s}}{\partial T}\right)_{P} \Delta T + \left(\frac{\partial s_{\rm s}}{\partial P}\right)_{T} \Delta P \approx \left(\frac{\partial s_{\rm s}}{\partial T}\right)_{\mu} \Delta T = \frac{c_{\mu,\rm s}}{T}\; \Delta T\,. \tag{15}$$

The approximate sign comes from the approximation (12). The corrections are of the relative order $\mathcal{O}(s_{\rm s}/s)$; they stem from the $\mu_{\rm s}$-term in (6). In the last step we introduced the specific heat c_μ at constant μ and the corresponding quantity $c_{\mu,\rm s}$.

We evaluate now the decisive equation (5). Taking into account (13) we see that $\partial_t(\rho s) = \rho\, \partial_t\, s$ and $\boldsymbol{\nabla}(\rho_{\rm s}\, s_{\rm s}\, \mathbf{u}_{\rm s}) = \rho_{\rm s}\, \mathbf{u}_{\rm s} \boldsymbol{\nabla} s_{\rm s}$. Using this, (9), (14) and (15) we obtain

$$\left(-\mathrm{i}\,\omega\,\rho\, \frac{c_\mu}{T} + \mathrm{i}\, k\, \rho_{\rm s}\, u_{\rm s}\, \frac{c_{\mu,\rm s}}{T}\right) \Delta T\, \exp(\mathrm{i}\,(k\,R\,\phi - \omega t)) = 0\,. \tag{16}$$

This shows that an excitation with the FP amplitudes (12) is a sound mode with velocity

$$c_6 \;=\; \frac{\omega}{k} \;=\; u_{\rm s}\, \frac{\rho_{\rm s}}{\rho}\, \frac{c_{\mu,\rm s}}{c_\mu}\,. \tag{17}$$

For $s_{\rm s} = 0$ the frequency ω becomes zero and the sixth sound reduces to a static FP gradient.

Eq. (17) is our central result: It connects the measurable velocity c_6 of the sixth sound with the unknown superfluid entropy. The quantities $\rho_{\rm s}/\rho$, c_μ and $u_{\rm s}$ are known or can be measured by standard methods. The sound velocity $c_6(T,P)$ determines thus $c_{\mu,\rm s}(T,P) = T\,(\partial s_{\rm s}/\partial T)_\mu$ and (using $s_{\rm s}(T_\lambda, P) = 0$) eventually $s_{\rm s}(T,P)$. Due to the weak pressure dependence of the entropies the approximations $c_\mu \approx c_P$ (= specific heat at constant pressure) and

$$s_{\rm s}(T,P) \approx \int_{T_\lambda}^{T} dT'\, \frac{c_{\mu,\rm s}(T',P)}{T'} \tag{18}$$

are valid within a few percent. Eq. (17) and (18) display the simple and rather direct connection between the sound velocity $c_6(T,P)$ and the superfluid entropy $s_{\rm s}(T,P)$.

We specify now in some detail an experiment by which $c_6(T,P)$ can be measured. The preparation of a persistent current in a ring-shaped superleak (Fig. 1) is a standard experiment [11]: Above T_λ the ring (radius R) is rotated with frequency $\omega_{\rm rot}$ around its symmetry axis. During rotation it is cooled to $T < T_\lambda$. After that the ring is stopped. The superfluid phase continues to rotate with $u_{\rm s} = R\,\omega_{\rm rot}$.

In this configuration one side of the ring is slightly heated. This will excite modes of the fourth and sixth sound propagating along the ring. The possible eigenmodes have wave numbers $k = n/R$ with $n = \pm 1, \pm 2, \ldots$. A major amplitude may be expected for the ground mode with $n = \pm 1$.

At a given time the ground mode of the sixth sound has a temperature variation of the form

$$\delta T(\phi) = A\,\cos(\phi) \tag{19}$$

accompanied by $\delta P \approx (s/v)\,\delta T$. The amplitude A depends on the excitation strength. The sixth sound may be conceived as an entropy transport process: Because of the temperature variation the persistent current causes a small *net entropy current density*

$$\delta j = \rho_{\rm s}\,u_{\rm s}\,\delta s_{\rm s} = \rho_{\rm s}\,u_{\rm s}\,\frac{c_{\mu,{\rm s}}}{T}\,\delta T(\phi)\;. \tag{20}$$

The current δj leads to a shift of the entropy density variation $\rho\,\delta s = \rho\,(c_\mu/T)\,\delta T(\phi)$ in ϕ-direction. This shift has the velocity $\delta j/\rho\,\delta s = c_6$. Consequently, the spatial variation (19) rotates with the frequency

$$\omega_{\rm FP} = \frac{c_6}{R}\;. \tag{21}$$

The direction of this rotation depends on the sign of $u_{\rm s}$ and $c_{\mu,{\rm s}}$.

At a given point of the ring one should be able to observe a *fountain pressure oscillation* with

$$\delta T(t) = A\,\cos(\omega_{\rm FP}\,t)\,\exp(-\Gamma_{\rm FP}\,t) \tag{22}$$

and $\delta P(t) \approx (s/v)\,\delta T$. This oscillation can be distinguished from the fourth sound because of $(\delta T/\delta P)_{\rm 4th} \ll (\delta T/\delta P)_{\rm 6th}$.

We estimate the damping coefficient $\Gamma_{\rm FP}$. The leading dissipation term on the r.h.s. of eq. (5) is $(m\kappa/T)\,\boldsymbol{\nabla}^2\,T(\mathbf{r},t)$ where κ is the heat conductivity. This becomes $-k^2(m\kappa/T)\,\Delta T\,\exp(\ldots)$ on the r.h.s. of (16). Consequently the frequency ω in (17) has an imaginary part. For $k = \pm 1/R$ we obtain

$$\Gamma_{\rm FP} = -{\rm Im}\,\omega = \frac{m\kappa}{R^2\rho\,c_\mu}\;. \tag{23}$$

As an example we insert the values $u_{\rm s} = 2\,{\rm cm/s}$ and $R = 2\,{\rm cm}$ used in an actual experiment [11]. For κ we take the upper value $0.05\,{\rm J/(m\,s\,K)}$ for He II cited in [3]. For $s_\mathrm{s}(T,P)$ we use the estimate of Ref. [8]. At $T = 2.15\,\mathrm{K}$ and saturated vapour pressure we obtain then

$$|\omega_\mathrm{FP}| \approx 0.01\ \mathrm{s}^{-1}\,, \qquad n_\mathrm{osc} = \frac{|\omega_\mathrm{FP}|}{2\pi\,\Gamma_\mathrm{FP}} \approx 16\,. \tag{24}$$

Here n_osc is the number of cycles after which the amplitude of the FP oscillation is reduced by a factor e. Eqs. (19) – (24) apply to the ground mode, $k = n/R$ with $n = \pm 1$. For higher modes one obtains $\omega_\mathrm{FP} \propto |n|$ and $n_\mathrm{osc} \propto 1/|n|$.

The numerical values given in (24) depend on the model assumptions used for the prediction [8] of s_s. The predicted s_s has a maximum at about $T = 2.05\,\mathrm{K}$; at this point ω_FP and n_osc vanish. The best region for starting the search for FP oscillations is the range between T_λ and $2.1\,\mathrm{K}$ or between $2\,\mathrm{K}$ and $1.8\,\mathrm{K}$ (for normal or saturated vapour pressure).

In the two-fluid model (4) – (6) neither the heat capacity nor the heat conductivity of the material (powder, containing ring) other than He II is taken into account. Therefore, our formulae have to be modified with respect to a specific experiment. The entropy current (20) is not affected by the surrounding material. The shift of the temperature variation (19) is, however, accompanied by that of the local entropy of the ring (He II plus powder plus container). Therefore, the specific heat c_μ in (16) and (17) has to be replaced by the effective specific heat of the ring:

$$c_\mu \longrightarrow c_\mathrm{ring}\,, \qquad \kappa \longrightarrow \kappa_\mathrm{ring}\,. \tag{25}$$

This applies also to the heat conductivity; the specific heat c_ring has to be related to one particle of the liquid. The necessity for $c_\mu \rightarrow c_\mathrm{ring}$ can be clearly seen from the discussion following eq. (19). A measurement of c_ring is necessary in order to determine s_s quantitatively from ω_FP. For example, $c_\mathrm{ring} = 2\,c_\mu$ would halve ω_FP. Favourably, the ring material should have a low heat capacity and a low heat conductivity. Furthermore, the observability can be improved by using a larger ring or a higher supervelocity (because of $n_\mathrm{osc} \propto R\,|u_\mathrm{s}|$).

We summarize: A non-vanishing superfluid entropy implies a new mode (sixth sound) in clamped He II: In the presence of a persistent current a FP gradient oscillates with the frequency ω_FP. The characteristic proportionality $\omega_\mathrm{FP} \propto u_\mathrm{s}$ and the moderate damping should allow the observation of the FP oscillation. If the sixth sound exists the superfluid entropy $s_\mathrm{s}(T,P)$ can be determined by measuring $\omega_\mathrm{FP}(T,P)$. If the sixth sound does not exist an experiment would yield a new upper limit (roughly $S_\mathrm{s}/S \leq 1\%/n_\mathrm{osc}$) for the superfluid entropy.

References

[1] F. I. Glick, J. H. Werntz, Jr., Phys. Rev. **178**, 314 (1969)

[2] A. Singsaas, G. Ahlers, Phys. Rev. **B 29**, 4951 (1984)

[3] S. J. Putterman, *Superfluid Hydrodynamics*, North Holland Publishing Comp., London 1974

[4] F. London, *Superfluids*, Vol. II, Wiley, New York 1954

[5] R. B. Dingle, Proc. Roy. Soc. (London) **A 62**, 648 (1949)

[6] F. London, Nature **141**, 643 (1938)

[7] R. P. Feynman, Phys. Rev. **91**, 1291 (1953)

[8] T. Fliessbach, Nuovo Cimento **D 13**, 211 (1991)

[9] R. Schaefer, *Hydrodynamik in Helium II mit nichtverschwindender superfluider Entropie*, Ph.D. thesis, Siegen 1993

[10] D. R. Tilley, J. Tilley, *Superfluidity and Superconductivity*, 3. edition, A. Hilger, Bristol 1990

[11] J. R. Clow, J. D. Reppy, Phys. Rev. **A 5**, 424 (1972)