Asymptotic temperature dependence of the superfluid density in liquid ^{4}He

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In a modified ideal Bose gas model we derive an expression for the temperature dependence of the superfluid fraction in liquid ^{4}He. This expression leads to a fit formula for the asymptotic temperature dependence that reproduces the data significantly better than comparable formulas.

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I. INTRODUCTION

The behavior of ^{3}He and ^{4}He liquid on one hand, and of the ideal Fermi and Bose gas on the other hand, strongly suggests that there is an intimate relation[1] between the λ transition in ^{4}He and the Bose-Einstein condensation (BEC) of the ideal Bose gas (IBG). Because of the neglect of the interactions one will not expect that the IBG reproduces all properties of liquid ^{4}He, in particular not those properties that are directly related to the interactions (like the specific heat or the compressibility). There are, however, some basic properties of liquid ^{4}He that may be explained[2,3] by the IBG (like the irrotational superfluid flow). We start by discussing a discrepancy between the IBG and liquid ^{4}He that is —as we will point out— disturbing in view of the suggested intimate relation between the BEC and the λ transition.

The critical behavior of the condensate fraction of the IBG is

$$\frac{\rho_0}{\rho} \sim |t|^{2\beta}, \qquad \beta = \frac{1}{2}\,, \tag{1}$$

where $t = (T - T_\lambda)/T_\lambda$ is the relative temperature; the IBG transition temperature is equated with that of the λ transition. The condensate fraction is commonly identified with the superfluid fraction; this identification explains[2,3] a number of experimental findings of which the most important one[4] is that a superfluid current has no vortices. In contrast to Eq. (1), the experimental superfluid fraction behaves like

$$\frac{\rho_\mathrm{s}}{\rho} \sim |t|^{2\nu}, \qquad \nu \approx \frac{1}{3}\,. \tag{2}$$

The suggested intimate connection between the BEC and the λ transition is in conflict with $\beta \neq \nu$. The values $\beta = 1/2$ and $\nu \approx 1/3$ imply $\rho_0 \ll \rho_\mathrm{s}$ just below the transition.

The standard solution of the conflict between Eqs. (1) and (2) appears to be the renormalization-group method. In this approach one starts from a Ginzburg-Landau ansatz for the free energy (or enthalpy) that leads to the critical exponent 1/2 for the order parameter. For the considered universality class the renormalization procedure yields then values near to 1/3 for the critical exponent of the order parameter. This may well serve as an explanation of the experimental value of $\nu \approx 1/3$ in Eq. (2) but it does not resolve the conflict between Eq. (1) and Eq. (2): A renormalization is appropriate for the Landau value $\beta = 1/2$ but not for the IBG value $\beta = 1/2$. The reason is that the IBG value is obtained by an exact evaluation of the partition sum. The exact evaluation of the partition sum implies a summation over arbitrarily small momenta (or, correspondingly, arbitrarily large distances). Therefore, the reasoning behind the renormalization procedure (analytic Ginzburg-Landau ansatz for *finite* regions or blocks, and subsequent transformation to larger and larger blocks) cannot be applied to the IBG free energy.

Moreover, the critical exponent $\beta = 1/2$ of the IBG cannot be changed within the IBG frame without destroying the mechanism leading to the BEC. The exponent $\beta = 1/2$ is characteristic for the IBG with the BEC phase transition.

We have now discussed two points: (i) We expect an intimate relation between the Bose-Einstein condensation and the λ transition. (ii) The IBG value $\beta = 1/2$ should be taken seriously (because it is a result of an exact evaluation of a partition sum, and because $\beta \neq 1/2$ is not compatible with the BEC mechanism). From these two points we conclude the following: The theoretical $\beta = 1/2$ in Eq. (1) and the experimental $\nu \approx 1/3$ in Eq. (2) are in conflict with each other.

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Within the frame of the ideal Bose gas model we propose to resolve this conflict by the assumption that *noncondensed particles move coherently with the condensate*. This means that we no longer identify the condensate with the superfluid fraction; the condensate is only part of the superfluid phase. A coherent motion can be described by multiplying the real single particle functions of noncondensed particles by the complex phase factor of the condensate. The superfluid density $\rho_{\rm s}$ is then made up by the condensate density ρ_0 plus the density $\rho_{\rm coh}$ of the coherently comoving, low momentum noncondensed particles. This concept leads to an expression and eventually to a fit formula for the temperature dependence of the superfluid density.

We will stick to the essential characteristic of the IBG (in particular the BEC) but introduce some modifications (for example, Jastrow factors) that are necessary for a realistic approach to liquid ^{4}He. This modified IBG is called *almost ideal Bose gas model* (AIBG). The AIBG has been introduced some years ago as an attempt to explain the (nearly) logarithmic singularity of the specific heat[5]. Some consequences of the decomposition $\rho_{\rm s} = \rho_0 + \rho_{\rm coh}$ have been discussed in Refs. 5 and 6. The present paper is devoted to the investigation of the temperature dependence of the superfluid density in this model. The necessary details of the underlying model, the AIBG, will be given below.

The form of the temperature dependence of the superfluid fraction is derived in Sec. II. This leads to a fit formula for the temperature dependence of ρ_s that is applied to experimental data and compared to other fit formulas (Sec. III). Sec. IV discusses the temperature dependence of the condensate density. Section V presents scaling arguments on the basis of an effective Ginzburg-Landau model; this includes a qualitative explanation of the coherent comotion of noncondensed particles and leads to restrictions for some of the parameters of the fit formula.

II. AIBG FORM OF THE SUPERFLUID FRACTION

A. Many-body wave function

Following Chester[7] we multiply the IBG wave function $\Psi_{\rm IBG}$ by Jastrow factors $F = \prod f_{ij}$,

$$\Psi = F\,\Psi_{\rm IBG} = \prod_{i<j}^{N} f_{ij}(r_{ij})\,\Psi_{\rm IBG}(\mathbf{r}_1, \ldots, \mathbf{r}_N; n_{\mathbf{k}}) \,. \tag{3}$$

We consider $i = 1, 2, ..., N$ atoms in a volume V. The occupation numbers $n_{\mathbf{k}}$ are parameters of the wave function; in physical quantities they are eventually replaced by their statistical expectation values $\langle n_{\mathbf{k}} \rangle$. The Jastrow factors take into account the most important effects of the realistic interactions; with a suitable choice for the f_{ij} (for example, $f_{ij}(r) = \exp[-(a/r)^b]$ with a and b determined by a variational principle[8]) the wave function (3) leads to a realistic pair-correlation function.

The IBG wave function $\Psi_{\rm IBG}$ in Eq. (3) is the symmetrized product of single-particle functions. We display this structure admitting at the same time a phase field Φ of the condensate:

$$\Psi = \mathcal{S}\, F \left[\,\exp(\,{\rm i}\,\Phi)\right]^{n_0} \prod_{\mathbf{k}\,\neq\,0} \left[\,\varphi_{\mathbf{k}}\,\right]^{n_{\mathbf{k}}} \,. \tag{4}$$

Here $\mathcal{S}$ denotes the symmetrization operator. The $\varphi_{\mathbf{k}}$ are the real single-particle functions of the noncondensed particles. The schematic notation $[\varphi_{\mathbf{k}}]^{n_k}$ stands for the product $\varphi_{\mathbf{k}}(\mathbf{r}_{\nu+1})\cdot\varphi_{\mathbf{k}}(\mathbf{r}_{\nu+2})\cdot\ldots\cdot\varphi_{\mathbf{k}}(\mathbf{r}_{\nu+n_k})$; this notation applies also to $[\exp(\,{\rm i}\,\Phi)]^{n_0}$. All n_0 condensed particles adopt the same phase factor $\exp(\,{\rm i}\,\Phi(\mathbf{r}))$ forming the macroscopic wave function

$$\psi(\mathbf{r}) = \sqrt{\frac{n_0}{V}}\;\exp\left[\,{\rm i}\,\Phi(\mathbf{r})\right] . \tag{5}$$

The phase field Φ describes the coherent motion of the condensate particles. (Actually, one has to construct a suitable coherent state[9]. This point is, however, not essential for the following discussion.) This motion is superfluid if the velocity $\mathbf{u}_{\rm s} = \hbar\,\nabla\Phi/m$ is sufficiently small.

Equations (4) and (5) are a well-known description[2] for a superfluid motion in the IBG. In this description the superfluid fraction $\rho_{\rm s}/\rho$ equals the condensate fraction $n_0/N = \rho_0/\rho$. The role of the Jastrow factors in this context will be discussed in Sec. II B.

In order to dissolve the discrepancy between Eqs. (1) and (2) we assume that *noncondensed particles move coherently with the condensate*. This is possible if noncondensed particles adopt the macroscopic phase of the condensate:

$$\Psi = \mathcal{S}\, F \left[\,\exp(\,{\rm i}\,\Phi)\right]^{n_0} \prod_{0\,<\,k\,\le\,k_{\rm coh}} \left[\,\varphi_{\mathbf{k}}\;\exp(\,{\rm i}\,\Phi)\,\right]^{n_k} \prod_{k\,>\,k_{\rm coh}} \left[\,\varphi_{\mathbf{k}}\,\right]^{n_k} \,. \tag{6}$$

We assume the phase ordering for all states with momenta below a certain coherence limit $k_{\rm coh}$. For the low lying states with $n_k \gg 1$ such phase ordering is relatively easy because it requires only a small entropy decrease. At this stage, $k_{\rm coh}$ should be considered as a model parameter. In Sec. V A the existence and the size of this coherence limit will be made plausible.

We evaluate particle current for the wave function (6):

$$\mathbf{j}_{\rm s}(\mathbf{r}, n_k) = \left\langle \Psi \left| \sum_{n=1}^{N} \hat{\mathbf{j}}_n + \text{c.c.} \right| \Psi \right\rangle = \frac{\rho}{N}\,\frac{\hbar}{m}\left(n_0 + \sum_{k<k_{\rm coh}}{}' n_k \right) \nabla\Phi\;. \tag{7}$$

The prime at the sum over the momenta $k < k_{\rm coh}$ means that the $k = 0$ contribution is excluded. In coordinate space, the current operator reads $\mathbf{j}_n = -\mathrm{i}\,\hbar\nabla_n/(2\,m) + \text{c.c.}$ It acts on all $\mathbf{r}_i$-dependences. Because of the added conjugate complex term all contributions from the real functions (the f_{ij} in F or the φ_k) cancel. The only surviving terms are those where $\mathbf{j}_n$ acts on the phase Φ.

For a superfluid motion with $\mathbf{u}_{\rm s} = \hbar\,\nabla\Phi/m$ and in the statistical average, $\mathbf{j}_{\rm s}$ of Eq. (7) equals $\rho_{\rm s}\,\mathbf{u}_{\rm s}$. We may then read off the superfluid fraction,

$$\frac{\rho_s}{\rho} = \frac{1}{N}\left(\langle n_0\rangle + \sum_{k<k_{\rm coh}}{}' \langle n_k\rangle \right) = \frac{\rho_0 + \rho_{\rm coh}}{\rho}\;. \tag{8}$$

This expression will be evaluated in Sec. II C.

The ansatz (6) leading to Eq. (8) shows in which way noncondensed particles may contribute to the superfluid density.

B. Condensate density

We discuss in some detail what is meant by the terminus "condensate density", in particular with respect to the Jastrow factors in Eqs. (4) or (6).

The *exact condensate density* may be defined by

$$\langle\Psi|\,\hat{\phi}^{+}(\mathbf{r})\,\hat{\phi}(\mathbf{r}')\,|\Psi\rangle \;\xrightarrow{|\mathbf{r}-\mathbf{r}'|\to\infty}\; \rho_0^{\rm exact}\;, \tag{9}$$

where Ψ is the exact many-body state and the $\hat{\phi}^{+}$ and $\hat{\phi}$ are single-particle creation and annihilation operators. For finite temperatures one has to take the statistical expectation value of $\rho_0^{\rm exact}$ (we do not introduce a different symbol).

For an IBG wave function $\Psi_{\rm IBG}$ the condensate density is given by $\rho_0^{\rm model} = n_0/V$. In the statistical average this this *model condensate density* becomes

$$\rho_0^{\rm model} = \frac{\langle n_0\rangle}{V}\;. \tag{10}$$

The exact many-body state in Eq. (9) may be approximated by Eq. (3), or by $\Psi \approx F$ for the ground state. In this case the relation between both condensate densities is well known: The model condensate fraction is *depleted*[10,11] by the Jastrow factors F, for example, from $\rho_0^{\rm model}/\rho = 1$ to $\rho_0^{\rm exact}/\rho \approx 0.1$ for $T = 0$.

The above calculation leading to Eq. (8) demonstrates the following point: In contrast to the density $\rho_0^{\rm model}$, *the current density* $\rho_0^{\rm model}\,\mathbf{u}_{\rm s}$ *is not depleted*. The reason is that in Eq. (7) all derivatives of the real Jastrow factors cancel (because of the added conjugate complex term).

On the basis of this point we arrive at the following statements about the role of the densities $\rho_0^{\rm model}$, $\rho_0^{\rm exact}$, and $\rho_{\rm s}$.

(a) Since the current density $\rho_0^{\rm model}\,\mathbf{u}_{\rm s}$ is not depleted we may identify $\rho_0^{\rm model}$ (and not $\rho_0^{\rm exact}$) with the square $|\psi|^2$ of the macroscopic wave function. Irrespective of the Jastrow factors we may use Eq. (5) as it stands. For a superfluid flow, the phase $\Phi(\mathbf{r})$ of the macroscopic wave function (5) fixes the velocity field $\mathbf{u}_{\rm s} = \hbar\,\nabla\Phi/m$. The basic relations for the superfluidity (like $\mathrm{curl}\,\mathbf{u}_{\rm s} = 0$ and the Feynman-Onsager quantization rule) are not affected by the Jastrow factors in the many-body wave function.

(b) The exact condensate density is a quantity of its own right. It is the density of the zero momentum particles in the liquid helium. Recently Wyatt[12] reported about a rather clear experimental evidence for this condensate. For a review about the attempts to determine $\rho_0^{\rm exact}$ experimentally we refer to Sokol[13].

(c) The assumption that noncondensed particles move coherently with the condensate is introduced by the step from Eq. (4) to Eq. (6). Again, this step does not alter the basic relations following from Eq. (5) (like $\mathbf{u}_{\rm s} = \hbar\,\nabla\Phi/m$, $\mathrm{curl}\,\mathbf{u}_{\rm s} = 0$ and the Feynman-Onsager quantization rule).

(d) For $T = 0$ the value $\rho_0^{\rm model}/\rho = 1$ yields $\rho_{\rm s}/\rho = 1$ [as we will see, $\rho_{\rm coh}$ in Eq. (8) contributes only in the vicinity of T_λ]. In contrast to this the connection between $\rho_0^{\rm exact}/\rho \approx 0.1$ with $\rho_{\rm s}/\rho = 1$ is less obvious. For $T \approx 0$ the value $\rho_0^{\rm model}/\rho \approx 1$ implies $\rho_{\rm s}/\rho \approx 1$. For describing $1 - \rho_{\rm s}/\rho$ quantitatively one must however include phonons. This is not done in Eqs. (4) or (6) because our primary object is the asymptotic temperature region.

We summarize this subsection: As far as the superfluid current is concerned the model condensate density is not depleted. The model condensate density is the fundamental constituent of the superfluid density.

In the following the model condensate density $\rho_0^{\rm model}$ will again be denoted by ρ_0 and called condensate density.

C. Superfluid density

We evaluate the expression (8) for the superfluid density. Our model assumes expectation values $\langle n_k \rangle$ that are of the IBG form,

$$\langle n_k \rangle = \frac{1}{\exp\left[(\,\epsilon_k - \mu)/k_{\rm B}T\right] - 1} \;=\; \frac{1}{\exp\left(x^2 + \tau^2\right) - 1}\;. \tag{11}$$

Here μ is the chemical potential, $\epsilon_k = \hbar^2 k^2/2m$ are the single-particle energies, and $k_{\rm B}$ is Boltzmann's constant. In the last expression we introduced the dimensionless quantities $\tau^2 = -\mu/k_{\rm B}T$ and

$$x = \frac{\lambda\,|\mathbf{k}|}{\sqrt{4\pi}}\,, \quad \text{with} \quad \lambda = \frac{2\pi\hbar}{\sqrt{2\pi m\, k_{\rm B}T}}\;. \tag{12}$$

The transition temperature of the IBG is given by the following condition for the thermal wave length λ:

$$\lambda(T_\lambda) = \big[\, v\,\zeta(3/2)\,\big]^{1/3}\,, \tag{13}$$

where $\zeta(3/2) = 2.6124$ denotes Riemann's zeta function. In applying our almost ideal Bose gas model (AIBG) to the real system we identify T_λ with the actual transition temperature. In the following we use the relative temperature

$$t = \frac{T - T_\lambda}{T_\lambda}\;. \tag{14}$$

We evaluate the condensate density:

$$\frac{\rho_0}{\rho} = 1 - {\sum}'\,\frac{\langle n_{\mathbf{k}} \rangle}{N} = 1 - (1+t)^{3/2}\,\frac{g_{3/2}(\tau)}{\zeta(3/2)}\;. \tag{15}$$

Riemann's generalized zeta function is given by $g_p(\tau) = \sum_1^\infty \exp(-n\,\tau^2)/n^p$, and $\zeta(p) = g_p(0)$. The chemical potential μ or, equivalently, τ may be expanded for $|t| \ll 1$:

$$\tau(t) = \sqrt{\frac{-\mu}{k_{\rm B}T}} = \begin{cases} a\,t + b\,t^2 + \ldots & (t > 0) \\ a'\,|t| + b'\,t^2 + \ldots & (t < 0) \end{cases}\;. \tag{16}$$

For $t > 0$ Eq. (15) with $\rho_0/\rho = 0$ yields $(1+t)^{3/2}\,g_{3/2}(\tau) = \zeta(3/2)$. This condition determines the temperature dependence of $\tau(t)$ and in particular the coefficients a, b, ..., for example $a = 3\,\zeta(3/2)/(4\pi^{1/2})$.

For $t < 0$ the IBG yields $\tau = 0$. In the AIBG we admit nonvanishing coefficients a', b',... in Eq. (16). This makes the expansion (16) more symmetric; it corresponds to a phenonemological gap between the condensate level and the noncondensed particles. A coefficient $a' \neq 0$ does not affect the BEC as the most important feature of IBG. It avoids, however, the divergence of the static structure factor[14] $S(k)$ for $k \to 0$ and greatly improves the unrealistic ($\propto T^{3/2}$) behavior of the specific heat. In view of the successful roton picture it is not too surprising that a gap is necessary for a quantitative description of the superfluid density (or of the specific heat). As we will see, a realistic description of liquid helium requires $a' \approx 3$; the next coefficient b' will not be needed.

From Eq. (15) and with Eq. (16) we obtain

$$\frac{\rho_0}{\rho} = f\,|t| + g\,t^2 + \ldots \qquad (t<0) \tag{17}$$

where

$$f = \frac{3}{2} + \frac{2\sqrt{\pi}\,a'}{\zeta(3/2)}\,. \tag{18}$$

We evaluate now the density of the comoving particles

$$\frac{\rho_{\rm coh}}{\rho} = {\sum_{k<k_{\rm coh}}}' \frac{\langle n_k\rangle}{N} = \frac{4\,(1+t)^{3/2}}{\sqrt{\pi}\,\zeta(3/2)} \int_0^{x_{\rm coh}} \frac{x^2\,dx}{\exp(x^2+\tau^2)-1}\,. \tag{19}$$

Using $y/[\exp(y)-1] = 1 - y/2 + y^2/12 \mp \ldots$ we obtain

$$\frac{\rho_{\rm coh}}{\rho} = \frac{4\,(1+t)^{3/2}}{\sqrt{\pi}\,\zeta(3/2)} \left(x_{\rm coh} - \tau\, \arctan\frac{x_{\rm coh}}{\tau} - \frac{x_{\rm coh}^3}{6} + \frac{x_{\rm coh}^5}{60} + \frac{x_{\rm coh}^3\,\tau^2}{36} \pm \ldots \right). \tag{20}$$

The convergence of this expression is excellent; for the actual parameter values and for $|t| \le 0.1$ the terms not shown are of the order 10^{-8}.

We have not yet specified the coherence limit $k_{\rm coh}$. For $|t| \ll 1$ we will find $\rho_{\rm s} \sim \rho_{\rm coh} \sim k_{\rm coh}$ for the superfluid density and $\rho_{\rm s}\,k_{\rm coh}^2 \sim k_{\rm coh}^3$ for the kinetic energy of the fluctuations. Requiring that this kinetic energy scales with the free energy $F \sim -\langle n_0\rangle^2 \sim -t^2$ yields

$$k_{\rm coh} \sim |t|^{2/3}\,. \tag{21}$$

This scaling argument will be presented in more detail in Sec. V.

Inserting Eq. (21) in Eq. (20) and using Eq. (17), the superfluid fraction contains the powers $|t|^{2/3}$, $|t|$, $|t|^{4/3}$, and so on:

$$\frac{\rho_s}{\rho} = \frac{\rho_0 + \rho_{\rm coh}}{\rho} = a_1\,|t|^{2/3} + a_2\,|t| + a_3\,|t|^{4/3} + \ldots\,. \tag{22}$$

D. AIBG assumptions

We summarize in which points the AIBG, the almost ideal Bose gas model, deviates from the IBG:

1. The IBG wave function $\Psi_{\rm IBG}$ is multiplied by Jastrow factors, $\Psi = F\,\Psi_{\rm IBG}$. This is a well-known approach[7].

2. By the symmetric expansion (16) we admit a gap between the condensed and the noncondensed particles. This modification preserves the most basic features of the IBG, in particular the BEC mechanism and the critical exponent $\beta = 1/2$.

3. The noncondensed single-particle states below the coherence limit $k_{\rm coh}$ adopt the macroscopic phase of the condensate. The leading exponent for the coherence limit $k_{\rm coh}$ is determined from a scaling argument.

III. FIT TO EXPERIMENTAL DATA

We compare the temperature dependence of our model expression for the superfluid density with experimental data. The model expression contains unknown parameters; it provides a fit formula for the data. It will turn out that this fit formula is significantly better than comparable fit formulas.

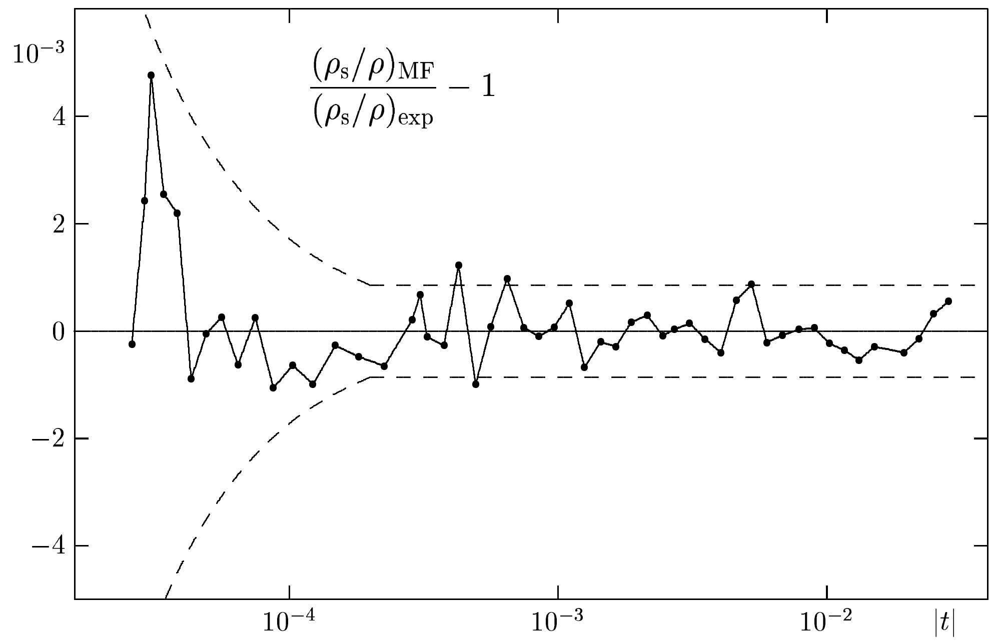

FIG. 1. The asymptotic model fit (MF), $(\rho_s/\rho)_{MF} = a_1\,|t|^{2/3} + a_2\,|t| + a_3\,|t|^{4/3}$, reproduces perfectly the experimental superfluid fraction $(\rho_s/\rho)_{exp}$. The solid points mark the relative deviations between the fit formula and the experimental values of Ref. 15. As a guiding line we draw straight lines between neighboring points. The broken line corresponds to two standard deviations.

A. Asymptotic temperature range

1973 data by Greywall and Ahlers

A restriction to the first three terms in the expansion (22) yields a three-parameter *model fit* (MF)

$$\frac{\rho_s}{\rho} = a_1\,|t|^{2/3} + a_2\,|t| + a_3\,|t|^{4/3} \qquad \text{(MF)}\ . \tag{23}$$

Figure 1 shows that the MF yields an excellent reproduction of the data by Greywall and Ahlers[15] for saturated vapor pressure. We used all data points with temperatures $|t| \le 0.03$. For a minor improvement of the fit we shifted the temperature values by -0.5×10^{-7}; this is well below the experimental uncertainty of $\delta t = 2\times10^{-7}$. The parameters of the fit shown in Fig. 1 are:

$$a_1 = 2.3233\,, \qquad a_2 = 1.0258\,, \qquad a_3 = -2.0065\ . \tag{24}$$

As an alternative we consider the *standard fit* (SF)

$$\frac{\rho_s}{\rho} = k\,|t|^{\xi}\left(1 + D\,|t|^{\Delta}\right) \qquad \text{(SF)}\,, \tag{25}$$

which is used by Greywall and Ahlers[15], and that is motivated by the renormalization-group theory. The fourth parameter Δ is often[15,16] set equal to 1/2 because the fit is not very sensitive to it. We will use the SF with $\Delta = 0.5$ as a three-parameter ansatz.

The fit parameters are found by minimizing the sum χ^2 of the quadratic deviations,

$$\chi^2 = \sum_{i=1}^{N_d} W\left[\left(\frac{\rho_s}{\rho}\right)_{fit} - \left(\frac{\rho_s}{\rho}\right)_{exp}\right]^2 = \sum_{i=1}^{N_d}\frac{1}{\sigma_{rel}^2}\left[\frac{(\rho_s/\rho)_{fit}}{(\rho_s/\rho)_{exp}} - 1\right]^2 . \tag{26}$$

Here N_d is the number of data points and σ_{rel} is the relative standard deviation. The standard deviation σ for $(\rho_s/\rho)_{fit} - (\rho_s/\rho)_{exp}$ is given by $W = 1/\sigma^2$. The dominant experimental error is that in the temperature. This leads to the weight[15] $W = |t|^{2/3}/\delta|t|^2$, where $\delta|t| = \max(2\times 10^{-7}, 10^{-3}\,|t|)$ is the temperature uncertainty. The corresponding $2\,\sigma_{\rm rel}$ line is shown in Fig. 1.

In the given form both MF and SF (with $\Delta = 0.5$) are three-parameter fits. We compare both fits by calculating their χ^2 ratio:

$$\frac{\chi^2_{\rm SF}}{\chi^2_{\rm MF}} \approx 8.8 \qquad (\text{data for } |t| \le 0.03)\ . \tag{27}$$

As seen from Fig. 1 the MF reproduces the experimental data ($\chi^2/N_{\rm d} \approx 1.10$). The large ratio (27) means that the SF does not reproduce the data in the considered temperature range.

We remark that the SF fits the data in the considerably smaller range $|t| \le 0.004$. This smaller range is used in Ref. 15, presumably because it was realized that the SF does not fit the data in the larger range. For a three-parameter fit the range $|t| \le 0.004$ appears to be rather small; we note that already a one-parameter fit ($a_1|t|^{2/3}$) reproduces the data within 2% in the relatively large range $|t| \le 0.08$.

We considered also the data at higher pressures by Greywall and Ahlers[15]. Here we found ratios $\chi^2_{\rm SF}/\chi^2_{\rm MF}$ between 1 and 2, and values of $\chi^2_{\rm MF}/N_{\rm d}$ in the range between 3.6 and 15. This means that the MF is only slightly better than the SF without yielding satisfactory fits. This is (at least partly) caused by jumps in the experimental data points. For example, compared to a smooth fit curve (SF or MF or any reasonable fit formula) there is a jump of more than ten standard deviations between the data points $(|t|,\, \rho_{\rm s}/\rho) = (0.001\,439\,1,\, 0.028\,144)$ and $(0.001\,263\,1,\, 0.025\,624)$ for $P = 7.27\,$bar.

1993 data by Goldner, Mulders and Ahlers

Newer measurements of the superfluid density are reported by Goldner et al.[16] and by Marek et al.[17]. We consider the data by Goldner et al. because these authors published an explicit data list.

The data[16] extend to about $|t| = 0.01$; all these data are used for the fits. Fig. 2 shows how the three-parameter MF reproduces these data. We discuss this result in a number of points:

1. Obviously the scatter of the data is generally larger than the estimated error (listed as $\delta\rho_{\rm s}/\rho$ in Ref. 16, and called σ in Fig. 2). There are several jumps of the size of ten standard deviations; the most dominant jump (between the values for $|t| = 0.00031910$ and $|t| = 0.00039793$) is about 30 times larger than the estimated error. This statement is basically independent of the fit formula used (see also Fig. 3). It is extremely unlikely that the actual superfluid fraction contains such jumps. The different sizes of the jumps restrict the possibility to discriminate between various fit formulas. This is also the reason why we considered first the older 1973 data by Greywall and Ahlers.

2. The three-parameter SF yields a significantly larger χ^2 value:

$$\frac{\chi^2_{\rm SF}}{\chi^2_{\rm MF}} \approx 2.7 \qquad . \tag{28}$$

3. Goldner et al.[16] used the following *extended standard fit* (ESF)

$$\frac{\rho_s}{\rho} = k_0\,|t|^{\xi}\left(1 + D\,|t|^{\Delta}\right)(1 + k_1\,|t|) \qquad (\text{ESF}) \tag{29}$$

   with $\Delta = 1/2$. Using the same parameters as in Ref. 16 we obtained $\chi^2_{\rm ESF}/\chi^2_{\rm MF} \approx 1.4$. This might appear as a small difference between MF and ESF. A comparison between Figs. 2 (MF) and 3 (ESF) shows, however, that the MF does a better job although it has one parameter less. Goldner et al.[16] noted that there is a serious discrepancy between the ESF and the data, in particular in the range $|t| \approx 10^{-5}$ to 10^{-6} (their Fig. 17). The comparison between Fig. 2 and 3 shows that this discrepancy is significantly smaller for our model fit. This improvement is not so evident in the χ^2 ratio because the χ^2 values are on a high level for any fit formula (due to the jumps).

4. Looking at the scatter of the data one might tentatively assume a standard deviation that is five times larger than the one assumed. Drawing then a new $2\,\sigma$ line the discrepancies in the range $|t| = 10^{-5}$ to 10^{-6} in Fig. 2 may be judged as not very significant. They may, however, hint at an unexplained structure in the temperature dependence of the superfluid fraction.

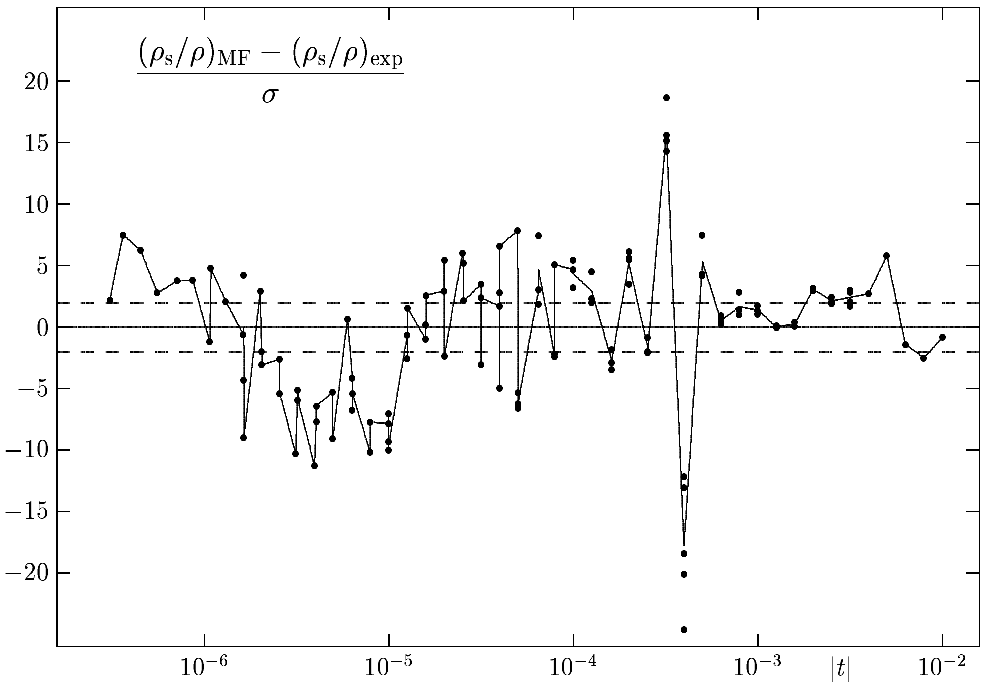

FIG. 2. The asymptotic model fit (MF), Eq. (23), is applied to the 1993 data of Ref. 16. The deviations between the fit and the data are given in units of the experimental error; the broken line corresponds to two standard deviations. For some temperatures there are several experimental ρ_s values (each leading to a full point); in this case the guiding line goes through the average value of the deviation. Sometimes the experimental temperature values are close together (but different); this results in apparently vertical pieces of the guiding line. From the figure it is obvious that (i) the statistical errors are considerably larger than the assumed σ, and (ii) the data contain several large jumps between neighboring points.

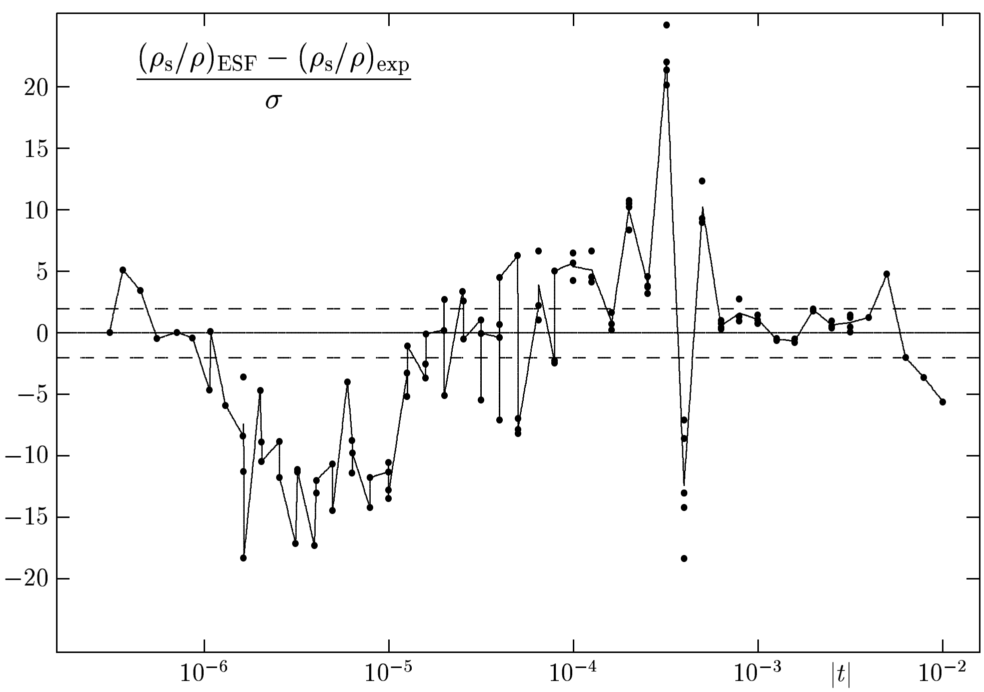

FIG. 3. The extended standard fit (ESF), Eq. (29), is applied to the 1993 data of Ref. 16. The presentation is the same as in Fig. 2. The systematic deviations between the fit and the data are significantly larger for this four-parameter ESF than for the three-parameter MF (Fig. 2).

B. Extension to lower temperatures

We apply the model expression for the superfluid fraction in the temperature range $1.2\,\mathrm{K} < T < T_\lambda$, where $|t|$ is no longer *much* smaller than 1. For this purpose we use the model expressions (15) and (19) for ρ_0 and $\rho_{\rm coh}$, respectively, and expand τ and $x_{\rm coh}$ (rather than $\rho_{\rm s}$ itself) into the relevant powers of $|t|$.

The expansion $\tau = a'\,|t| + b'\,t^2 + \ldots$ may be broken off after the first term because $\tau \neq 0$ corresponds to a gap and leads to an exponential decrease $[\propto \exp(-\tau^2)]$ of the noncondensed contribution in Eqs. (15) and (19). Therefore, the noncondensed contributions become rather small before the next terms in the expansion for τ contribute significantly.

As far as the coherence limit $k_{\rm coh}$ is concerned we have no information about the continuation of Eq. (21) into an expansion. In view of the success of Eq. (23) we will certainly not admit exponents that would violate the form (22). In accordance with Eq. (22) we may admit the form $x_{\rm coh} = x_1\,|t|^{2/3} + x_2\,|t| + x_3\,|t|^{4/3} + \ldots$. This expansion may be broken off, too, because $\rho_{\rm coh}$ of Eq. (19) is damped exponentially $[\propto \exp(-\tau^2)]$ for increasing $|t|$. Including the terms with the parameters x_1, x_2, and x_3 preserves the variability for the parameters a_1, a_2, and a_3 in Eq. (22).

Due to the exponential damping of the noncondensed contributions a cut in the expansions for τ and $x_{\rm coh}$ leads much further than a cut in the expansion for $\rho_{\rm s}/\rho$ itself. In this way we arrive at the following *unified model fit* (UMF) formula:

$$\frac{\rho_{\rm s}}{\rho} = 1 - (1+t)^{3/2}\,\frac{g_{3/2}(\tau)}{\zeta(3/2)} + \frac{4\,(1+t)^{3/2}}{\sqrt{\pi}\,\zeta(3/2)} \int_0^{x_{\rm coh}} \frac{x^2\,dx}{\exp(x^2+\tau^2)-1} \qquad \text{(UMF)} \tag{30}$$

with

$$\tau = a'\,|t|\,, \qquad x_{\rm coh} = \max\Big(0,\ x_1\,|t|^{2/3} + x_2\,|t| + x_3\,|t|^{4/3}\Big)\ . \tag{31}$$

As we will see, this formula provides a unified description of the asymptotic region as well as of the less asymptotic (the "roton") region.

The parameters x_1, x_2, and x_3 are related to the a_1, a_2, and a_3 in Eq. (22) and essentially fixed by the asymptotic region. We have restricted $x_{\rm coh}$ explicitly to non-negative values because the expression $x_1\,|t|^{2/3}+x_2\,|t|+x_3\,|t|^{4/3}$ might become negative for larger $|t|$ values [where, however, the density $\rho_{\rm coh}$ tends to zero anyway because the exponential decrease $\propto \exp(-\tau^2)$; see also Fig. 5].

For a fit in the range $1.2\,\mathrm{K} < T < T_\lambda$ we combined the data by Greywall and Ahlers[15] for $|t| < 0.04$ and that by Clow and Reppy[18] (run IV) for $|t| > 0.04$. At $|t| = 0.04$ both data sets are compatible with each other. The systematic errors and the deviation due to slightly different pressures (roughly 1% between saturated vapor or normal pressure) just happen to cancel each other. Clow and Reppy[18] remark that their "values of $\rho_{\rm s}/\rho$ have a scatter of about 1/2%"; we interpreted this as $\sigma_{\rm rel} = 0.005$ for our fit [i.e. for the minimization of Eq. (26)].

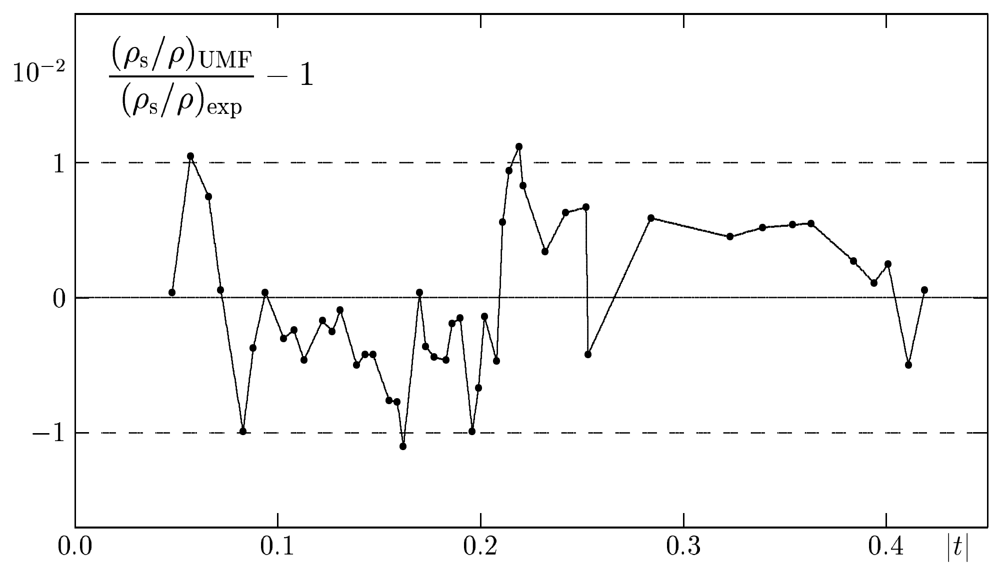

FIG. 4. The unified model fit (UMF), Eq. (30), is used to reproduce the experimental superfluid fraction $(\rho_{\rm s}/\rho)_{\rm exp}$ in the temperature range from T_λ to 1.2 K. For $|t| \leq 0.04$ we use the Greywall-Ahlers data (Ref. 15), for $|t| > 0.04$ that of Clow and Reppy (Ref. 18). The figure depicts the deviations for the range $|t| > 0.04$; for $|t| < 0.04$ the deviations are quite similar to that shown in Fig. 1. The broken line corresponds to two standard deviations.

A fit of the combined data[15,18] leads to a result that is quite similar to Fig. 1 for $|t| < 0.04$ and that is shown in Fig. 4 for $|t| > 0.04$. The fit parameters are:

$$a' = 3.0380\,, \quad x_1 = 2.6998\,, \quad x_2 = -0.8063\,, \quad x_3 = -3.9631 \qquad \text{(UMF)}. \tag{32}$$

Alternatively we may use the parameter f, Eq. (18), and calculate the parameters a_1, a_2, and a_3 following from the asymptotic expansion of Eq. (30):

$$f = 5.6225\,, \quad a_1 = 2.3323\,, \quad a_2 = 0.8035\,, \quad a_3 = -0.4704\,. \tag{33}$$

If an expansion is broken off as in Eq. (23) the last term tries effectively to simulate the missing terms. Since the UMF supplies higher-order terms it is not surprising that the last coefficients in Eqs. (24) and (33) are quite different.

Alternatively we used the data by Tam and Ahlers[19] that extend, however, only down to 1.5 K. This yields similar parameter values.

The standard fit for temperatures above 1 K but excluding the asymptotic region is the two-parameter *roton fit* (RF),

$$\frac{\rho_{\rm s}}{\rho} = \frac{A}{\sqrt{T}}\,\exp\left(-\frac{\Delta}{k_{\rm B}T}\right) \qquad \text{(RF)}\,. \tag{34}$$

Using the data of Ref. 18 (run IV) we obtain

$$\frac{\chi^2_{\rm RF}}{\chi^2_{\rm UMF}} \approx 4 \qquad \text{for } 1.2\,\text{K} < T < 2.07\,\text{K}\,. \tag{35}$$

This ratio is reduced to 2 if we restrict the temperature by $T < 2\,$K. These ratios imply that the unified model expression is quite good for intermediate temperatures, too.

The RF is based on Landau's quasiparticle model that cannot be extended to T_λ without loosing its physical basis. The standard description for the range $1.2\,\text{K} < T < T_\lambda$ would be a combination of the SF (25) and the RF (34). In contrast to this, our model provides a unified fit (30) in this range. Although containing one parameter less (than the combination of SF and RF) this unified fit is superior to the standard description.

As already mentioned, the expansion (16) implies a gap between the condensed and noncondensed particles. This gap appears to be essential for the reproduction of the data in the intermediate range $T \gtrsim 1\,$K. This gap should in some way be related to the roton gap Δ. This relation cannot be expected to be simple and obvious because one gap belongs to a model (Landau) for $T \ll T_\lambda$ and the other to a model (AIBG) for $T \sim T_\lambda$. We note that our gap vanishes for $T \to T_\lambda$, and that the roton concept becomes less sharp for increasing temperature (for $T = 1\,$K the widths of roton states are already comparable to their energies).

For $T \ll T_\lambda$ Landau's quasiparticle model is, of course, the right model. The model fit (30) yields still reasonable values for $\rho_{\rm s}/\rho$ but it must fail in the quantitative reproduction of $1 - \rho_{\rm s}/\rho$ because the phonons are not described by the wave function (6).

IV. CONDENSATE DENSITY

The unified model fit, Eq. (30) with Eq. (32), defines the decomposition of the superfluid density into the condensate density and the coherently comoving density. The temperature dependence of this decomposition is displayed in Fig. 5. In this section we discuss in particular the temperature dependence of the condensate density.

The contribution of $\rho_{\rm coh}$ is decisive near T_λ but negligible for lower temperatures. The comoving density $\rho_{\rm coh}$ carries some entropy because it does not correspond to a single quantum state. This entropy content is quite small because it is due to the lowest single-particle states with $\langle n_{\mathbf{k}}\rangle \gg 1$. It is below the present experimental limits but should be detectable; for these points we refer to Refs. 5 and 6.

As shown in Sec. III B, the expression $\tau = a'\,|t|$ works quite well for fitting the data down to about 1.2 K. Inserting $\tau = a'\,|t|$ in Eq. (15) yields

$$\frac{\rho_0}{\rho} = 1 - (1+t)^{3/2}\,\frac{g_{3/2}(a'\,|t|)}{\zeta(3/2)}\,. \tag{36}$$

Using a' of Eq. (32), this temperature dependence is shown by the dashed line in Fig. 5.

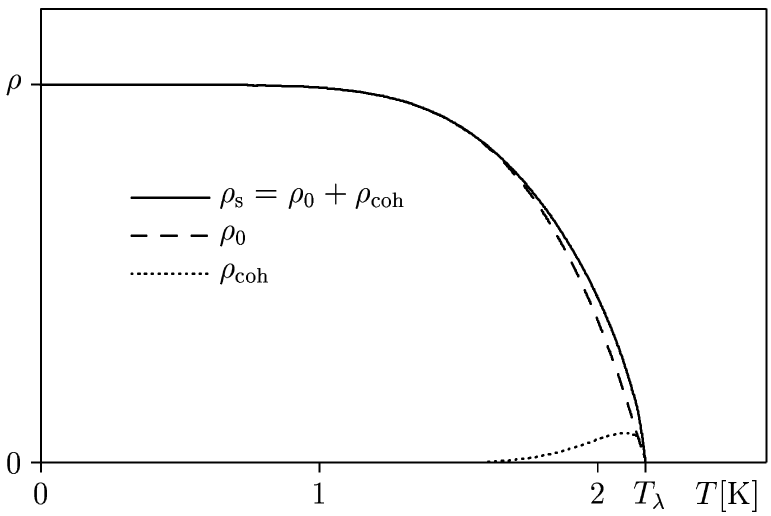

FIG. 5. Decomposition of the superfluid density ρ_s into the (model) condensate density ρ_0 and the coherently comoving density ρ_{coh} as a function of the temperature T. For $T \to T_\lambda$ the comoving density $\rho_{coh} \propto |t|^{2/3}$ is the dominant contribution to the superfluid density. The full line represents the model expression for ρ_s. In the given scale, this curve coincides with the experimental superfluid density for all temperature values.

The asymptotic expansion of Eq. (36) reads $\rho_0/\rho \sim f|t|$, Eq. (17), where

$$f = \frac{3}{2} + \frac{2\sqrt{\pi}\, a'}{\zeta(3/2)} \approx 5.6\,. \tag{37}$$

The numerical value is taken from Eq. (33). We found that

$$\frac{\rho_0}{\rho} \approx 1 - \left(\frac{T}{T_\lambda}\right)^f. \tag{38}$$

may be used as an approximation for Eq. (36). The maximum relative difference between Eqs. (38) and (36) is about 2%. For $T \to T_\lambda$ both expressions, (38) and (36), yield $\rho_0/\rho \sim f|t|$.

The right-hand side of Eq. (38) is an old fit formula for the superfluid fraction ρ_s/ρ (for example, Fig. 27 of Ref. 2). In the framework of our model, this historic fit formula may be interpreted as the approximation $\rho_s \approx \rho_0$. The obvious shortcomings of Eq. (38) as an approximation for ρ_s/ρ are the following: (i) The neglect ρ_{coh} leads to a qualitatively wrong asymptotic behavior (difference between the full and the dashed line in Fig. 5). (ii) The step from Eq. (36) to Eq. (38) as well as the use of $\tau = a'|t|$ make the expression an approximate one already for ρ_0/ρ. (iii) For small temperatures $1 - \rho_s/\rho = (T/T_\lambda)^f$ is quantitatively wrong (because the phonons have not been taken into account).

We consider once more the exact condensate fraction ρ_0^{exact}/ρ introduced in Sec. II B. We denote its value at $T = 0$ by n_c. Assuming that the depletion of the condensate (from 1 to $n_c \approx 0.1$) is temperature independent we obtain

$$\frac{\rho_0^{exact}}{\rho} \approx n_c\, \frac{\rho_0}{\rho} \approx n_c \left(1 - \frac{T^f}{T_\lambda^f}\right). \tag{39}$$

as an approximate expression for the temperature dependence of the exact condensate fraction. The experimental temperature dependence is given in Fig. 2 of Snow et al.[20]. Within the relatively large experimental uncertainties the expression (39) agrees with the data.

V. EFFECTIVE GINZBURG-LANDAU MODEL

In our approach, the macroscopic wave function (5),

$$\psi(\mathbf{r}) = \sqrt{\frac{n_0}{V}}\, \exp\left[\mathrm{i}\,\Phi(\mathbf{r})\right] = \sqrt{\rho_0}\, \exp\left[\mathrm{i}\,\Phi(\mathbf{r})\right], \tag{40}$$

plays the role of the order parameter. We investigate the free energy as a function of this order parameter.

A. Coherence limit

We start by presenting a qualitative argument for the existence and the meaning of the coherence limit $k_{\rm coh}$.

The macroscopic wave function ψ may contain equilibrium and nonequilibrium excitations. A superfluid motion with $\mathbf{u}_{\rm s} = (\hbar/m)\nabla\Phi$ is a nonequilibrium excitation. At finite temperatures, there are thermal fluctuations of the order parameter, i.e., equilibrium excitations. We consider the average momentum of these fluctuations,

$$k_{\rm fluct} = \overline{|\nabla\Phi|}\;. \tag{41}$$

The bar denotes the statistical average. The momentum $k_{\rm fluct}$ will be a function of the temperature. It is related to the correlation length $\xi \approx 1/k_{\rm fluct}$.

The single-particle states are described by real functions $\varphi_{\mathbf{k}}$ in Eq. (4). We consider the possibility of phase fluctuations for a low-lying state with $n_{\mathbf{k}} \gg 1$, too. After the replacement $\varphi_{\mathbf{k}} \longrightarrow \varphi_{\mathbf{k}} \exp(\mathrm{i}\,\Phi_{\mathbf{k}})$ in Eq. (4) these fluctuations may be described by the fields $\Phi_{\mathbf{k}}(\mathbf{r})$.

Let us first assume that the additional phases vanish, $\Phi_{\mathbf{k}} = 0$. In this case, the average kinetic energy $\hbar^2 k_{\rm fluct}^2/2m$ of a condensed particle would exceed that of a noncondensed particle with $k < k_{\rm fluct}$. The energy sequence of the single-particle states is, however, a prerequisite of the BEC; the condensate must be formed by the particles with the lowest energy. In order to preserve the energy sequence of the low-lying states we require the phase ordering

$$\Phi_{\mathbf{k}}(\mathbf{r}) = \Phi(\mathbf{r}) \quad \text{for} \quad k \le k_{\rm fluct}\;. \tag{42}$$

This argument does not apply to the states with higher momenta. By this qualitative argument we obtain the many-body wave function (6) with

$$k_{\rm coh} = k_{\rm fluct}\;. \tag{43}$$

B. Free energy

The statistical expectation value $\rho_0 \sim |t|$ can be obtained by minimizing the common Landau energy $F_{\rm L}/V = R\,t\,|\psi|^2 + U\,|\psi|^4$ (with regular coefficients R and U). The fluctuation term $F_{\rm fluct}/V = (\hbar^2/2m)\,|\nabla\psi|^2$ equals the kinetic energy density $\rho_0\,\mathbf{u}^2/2$ of the condensate only; here $\mathbf{u} = (\hbar/m)\nabla\Phi$. The phase coherence assumed in Eq. (6) implies that $\rho_0\mathbf{u}^2$ must be replaced by $\rho_{\rm s}\mathbf{u}^2$. This leads to the following *effective* Ginzburg-Landau ansatz

$$\frac{F_{\rm GL}}{V} = \frac{F_{\rm fluct} + F_{\rm L}}{V} = \frac{\hbar^2}{2m}\,\frac{\rho_s}{\rho_0}\,|\nabla\psi|^2 + R\,t\,|\psi|^2 + U\,|\psi|^4\;. \tag{44}$$

Assuming that the leading exponent of $x_{\rm coh}$ is not greater than 1, Eq. (20) yields

$$\rho_{\rm coh} \sim x_{\rm coh}\;. \tag{45}$$

The equilibrium fluctuation term becomes then

$$F_{\rm fluct} \propto \rho_s\,k_{\rm fluct}^2 = (\rho_0 + \rho_{\rm coh})\,k_{\rm coh}^2 \sim \rho_0\,x_{\rm coh}^2 + x_{\rm coh}^3\;. \tag{46}$$

The asymptotic form of the Landau part of the free energy behaves like

$$F_{\rm L} \propto R\,t\,\rho_0 + U\,\rho_0^2 \sim t^2\;. \tag{47}$$

We require now scaling invariance. This means that $F_{\rm fluct}$ must have the same leading $|t|$ dependence as $F_{\rm L}$, i.e., $\rho_0\,x_{\rm coh}^2 + x_{\rm coh}^3 \sim t^2$. From $\rho_0\,x_{\rm coh}^2 \sim t^2$ we would obtain $x_{\rm coh} \sim |t|^{1/2}$ and $x_{\rm coh}^3 \sim |t|^{3/2}$ in contradiction to the scaling assumption. Therefore, scaling requires $x_{\rm coh}^3 \sim |t|^2$ or

$$k_{\rm coh} \sim |t|^{2/3}\;. \tag{48}$$

This implies $\rho_s \sim \rho_{\rm coh} \sim |t|^{2/3}$ for the superfluid density and $\xi \approx 1/k_{\rm fluct} \sim |t|^{-2/3}$ for the correlation length.

The mass coefficient $\rho_s/\rho_0 \sim |t|^{-1/3}$ in Eq. (44) is singular. Ginzburg and Sobyanin[21] have introduced a comparable effective Ginzburg-Landau model with nonanalytic coefficients, too. In Ref. 21 the nonanalytic coefficients (like $|t|^{4/3}$ for the $|\psi|^2$ term) are phenomenologically introduced in order to reproduce the right critical exponents.

The divergent mass coefficient $\rho_s/\rho_0 \sim |t|^{-1/3}$ damps the critical fluctuations such that Eq. (44) becomes scaling invariant. In this sense, the model (44) has properties similar to the common Ginzburg-Landau ansatz in $d = 4$ dimensions. This means that Eq. (44) might be used down to $|t| = 0$ and that the critical exponent of ρ_s might be indeed exactly 2/3. This possibility is supported by the excellent fit obtained for Eq. (23).

C. Further scaling restrictions

The equilibrium Landau free energy contains integer powers of t only:

$$F_{\rm L} \sim \ldots t^2 + \ldots t^3 + \ldots t^4 + \ldots \,. \tag{49}$$

The asymptotic form of the superfluid density (22) is compatible with the expansion $x_{\rm coh} \propto x_1\,|t|^{2/3} + x_2\,|t| + x_3\,|t|^{4/3} + \ldots$. In the fluctuation term this expansion will, however, in general lead to noninteger exponents:

$$F_{\rm fluct} \propto \rho_{\rm s}\, k_{\rm coh}^2 = (\rho_0 + \rho_{\rm coh})\, k_{\rm coh}^2 \sim \ldots t^2 + \ldots |t|^{7/3} + \ldots |t|^{8/3} + \ldots t^3 + \ldots \,. \tag{50}$$

Scaling for Eq. (44) implies also that the amplitudes of the nonanalytic terms vanish. This condition yields relations between the expansion parameters a', b', … and x_1, x_2, x_3, … that may also be expressed by the coefficients a_i in Eq. (22). The condition of a vanishing amplitude of the $|t|^{7/3}$ term can be evaluated straightforwardly and yields

$$x_2 = -\frac{\sqrt{\pi}\,\zeta(3/2)}{8} \approx -0.58 \quad \text{or} \quad a_2 = 1\,. \tag{51}$$

These theoretical values compare well with the fitted values given in Eqs. (24) or (33).

The condition of a vanishing amplitude of the $|t|^{8/3}$ term yields

$$x_3 = \frac{\pi\,[\zeta(3/2)]^2}{64\,x_1} - \frac{a'^2}{3\,x_1} \tag{52}$$

and a corresponding expression for a_3. These relations are not fulfilled by the parameter values found in the fits. The reason is probably the following: The expansions (22) and (31) are cut after the $|t|^{4/3}$ term. In a fit it is then in particular the last term that tries to simulate the neglected terms.

VI. CONCLUDING REMARKS

We have modified the IBG in such a way that it might be applied to liquid helium. We summarize the novel views and main results of our approach.

1. The IBG value $\beta = 1/2$ for the critical exponent of the condensate should be taken seriously. It is not subject to renormalization because it results from a calculation that already includes a summation over arbitrarily large lengths, and it is essential for the BEC mechanism.

2. The model condensate contributes fully to the superfluid density; it is not depleted by the Jastrow factors.

3. In order to reproduce the critical exponent $\nu \approx 1/3$ of the superfluid density we have assumed that noncondensed particles below a certain momentum $k_{\rm coh}$ move coherently with the condensate. The coherence limit $k_{\rm coh}$ has been made plausible in Sec. V A.

   The contribution of noncondensed particles to the superfluid density offers a solution of the so-called macroscopic problem[22] of liquid helium. This problem reads as follows: If the superfluid density corresponds to single quantum state ($\rho_{\rm s} \propto |\psi|^2$) then the approach to an equilibrium state [with $\rho_{\rm s} = \rho_{\rm s}(T)$] cannot be understood.

4. We have derived a fit formula for the temperature dependence of the superfluid density. This fit formula reproduces the data significantly better than comparable expressions. This feature as well as qualitative scaling arguments suggest that the critical exponent ν of the superfluid density might be exactly equal to 2/3.

5. The temperature dependence of the decomposition of superfluid density into the model condensate density and the coherently comoving density is given. A simple formula for the temperature dependence of the depleted condensate density is presented.

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