Phase ordering at the lambda transition in liquid ^{4}He

Torsten Fliessbach*

Fachbereich Physik, University of Siegen

57068 Siegen, Germany

Abstract

A modified ideal Bose gas model is proposed as an approach for liquid helium at the lambda point. The decisive modification of the ideal Bose gas model is the use of phase ordered single particle functions. The entropy due to this phase ordering is calculated. Its statistical expectation value yields a logarithmic singularity of the specific heat.

PACS number: 67.40.–w

---

*Electronic address: fliessbach@physik.uni–siegen.de

1   Introduction

As suggested by London [1] and substantiated by Feynman [2] there is an intimate relation between the lambda transition in ^{4}He and the Bose-Einstein condensation of the ideal Bose gas (IBG). The IBG explains [3] some basic properties of liquid ^{4}He like for example the irrotational superfluid flow. We propose a modification of the IBG that leads to a realistic expression for the specific heat.

The single particle functions of the IBG are of the form

$$\varphi_{\mathbf{k}} \propto \exp(\mathrm{i}\,\mathbf{k} \cdot \mathbf{r}) = \exp[\mathrm{i}\,(k_1 x + k_2 y + k_3 z)]. \tag{1}$$

Without changing the kinetic energy we may alternatively consider the single particle functions

$$\varphi_{\mathbf{k}}^{\mathrm{p.o.}} \propto \exp[\mathrm{i}\,(k_2 y + k_3 z)]\,\sin(qx + \phi), \tag{2}$$

where $q = k_1$. For an arbitrary direction (chosen here as the x direction) the functions $\varphi_{\mathbf{k}}^{\mathrm{p.o.}}$ have a specific phase ϕ; they are *phase ordered* (p.o.). This phase ordering implies the following correlation: For $k_1 = k_1' = q$, the probability densities $|\varphi_{\mathbf{k}}^{\mathrm{p.o.}}|^2$ and $|\varphi_{\mathbf{k}'}^{\mathrm{p.o.}}|^2$ have common minima and maxima. That means that two particles with the same momentum in one extinguished direction have an additional positive spatial correlation. No such correlations are present for the single particle functions (1).

The phase ordering leads to a correlation energy of the form

$$E_{\mathrm{corr}} \propto \sum_q \nu_q^2 \quad \text{with} \quad \nu_q = \sum_{k_2,\,k_3} n_{\mathbf{k}}. \tag{3}$$

Here $n_{\mathbf{k}}$ is the number of atoms with momentum $\mathbf{k} = (q, k_2, k_3)$ and ν_q is the number of atoms with common momentum q in x direction. In the group of ν_q atoms each one has an extra correlation with the other atoms; this results in a ν_q^2 contribution to the energy. This correlation energy has been investigated in detail in Ref. [4].

We will use the IGB many-body wave functions together with Jastrow factors as proposed by Chester [5]. The effective interaction on the model level is then *attractive* (the hard core is cut out by the Jastrow factors). This means that the energy E_{corr} is negative and that the correlations are favoured by the condition of minimal free energy.

For a *finite* correlation energy one must assume that the phase ordering is *local*. The phase correlations of the single particle functions (2) could be enforced by physical boundary conditions for the considered macroscopic volume V (instead

of periodic boundary conditions for the single particle functions (1)). In this case the correlations are a surface effect implying $E_{\mathrm{corr}} \propto V^{2/3}$, and $E_{\mathrm{corr}}/V \to 0$ for $V \to \infty$. Assuming *local* phase ordering introduces a parameter V_0 that defines the finite range over which the single particle functions are correlated. The directions (x direction in Eq. (2)) of phase ordering are in general different in various subvolumes. The statistical average over these subvolumes ensures the isotropy of the bulk liquid. Evaluating the energy (3) with the IBG expectation values $\langle n_{\mathbf{k}} \rangle$ yields a logarithmic singularity [4], i.e., $\langle E_{\mathrm{corr}} \rangle \propto -t \ln|t|$, where t is the relative temperature. Adjusting the amplitude of this singularity to the experimental value fixes the model parameter V_0 (leading to $V_0/v \approx 10^5$, where v is the volume per atom).

It is the aim of this paper to calculate the entropy change due to the assumed local phase ordering. This calculation is independent of the previous determination of the correlation energy (3). In order to calculate the phase ordering entropy we establish a relation between the functions (2) and coherent states; this relation is analogous to Anderson's connection [6] between the condensate wave function and locally coherent states. For coherent states the phase variances are given by $\Delta\phi_{\mathbf{k}} = 1/(2\sqrt{n_{\mathbf{k}}})$. Restricting the phases by $\Delta\phi_{\mathbf{k}}$ yields the phase ordering entropy $S_{\mathrm{p.o}}(n_{\mathbf{k}})$. Using the occupation numbers $\langle n_{\mathbf{k}} \rangle$ of the IBG form we find logarithmic singularity for the thermodynamic entropy, $\langle S_{\mathrm{p.o}} \rangle \propto -t \ln|t|$. For the specific heat per particle, $c = -A \ln|t|$, we obtain the result $A = 3\,k_{\mathrm{B}}/2$, where k_{B} is Boltzmann's constant.

As a theory of the lambda transition, the presented approach is a modest attempt only. Instead of determining exact many-body states we make a guess of the relevant correlations. Instead of evaluating the partition sum we use the IBG expectation values (arguing that the correlation effects are of minor influence on the occupation of the individual levels). The determination of the exact many-body states *and* the exact evaluation of the partition sum is, of course, not feasible. One might, however, try to find a suitable variational ansatz for the many-body states for which the free energy can be calculated and minimized (like it may be done for BCS states). The present evaluation of the entropy together with the previous evaluation of the correlation energy (3) are a first step towards such an approach. As it stands, we present a phenomenological many-body model that yields a promising result, namely a logarithmic singularity with a sensible strength.

As a many-body approach for the lambda transition the presented model lies outside the main stream of realistic models for liquid helium. There are several well-known and thoroughly studied microscopic many-body approaches (for an overview see Ref. [7]), like diagrammatic (Green's function) approaches [8],

variational methods (in particular the correlated basis function method and its generalizations [9, 10]), and Monte Carlo calculations. As far as the approaches are based on elementary excitations they are restricted to low temperatures. For temperatures around the lambda point, there are successful applications of path-integral Monte Carlo simulations [11, 12] and of the variational approach [13]. These approaches do, however, not yield analytic and realistic expressions for the asymptotic behavior. What is nowadays widely considered as *the* theory of the lambda transition and its asymptotic properties is the renormalization group theory [14]. In contrast to the model presented here, this theory is not a many-body description.

Several authors [15, 16, 17, 18] have considered coherent states in the description of liquid helium, too. Their investigations concern mostly the connection between the condensate and coherent states as in Anderson's paper [6]. The local phase ordering that we consider is not specifically related to any of these other investigations.

2 Phase ordering entropy

2.1 Many-body states

Chester [5] studied the density matrix for many-body wave functions of the form $\Psi = \mathcal{S} F \Psi_{\text{IBG}}$. Here S denotes the symmetrization operator, $F = \prod_{i,j} f(r_{ij})$ is the Jastrow factor, $\Psi_{\text{IBG}}(\mathbf{r}_j, n_{\mathbf{k}})$ describes the IBG state, and $\mathbf{r}_j$ defines the positions of the helium atoms. Using hard sphere factors for $f(r)$, Chester obtained the IBG values $\langle n_{\mathbf{k}} \rangle_{\text{IBG}}$ for the average occupation numbers $\langle n_{\mathbf{k}} \rangle$. McMillan [19] used the ansatz $\Psi = \mathcal{S} F$ for the ground state; for realistic atom-atom interactions he determined the function $f(r)$ by minimizing the energy. This yields a realistic pair correlation function and a realistic binding energy.

We follow basically Chester's approach replacing, however, the single particle functions (1) by the phase ordered ones (2). The many-body wave function is then of the form

$$\Psi = \mathcal{S} F \prod_{j=1}^{N} \varphi_{\mathbf{k}_j}^{\text{p.o.}}(\mathbf{r}_j) = \mathcal{S} F \prod_{\mathbf{k}} [\varphi_{\mathbf{k}}^{\text{p.o.}}]^{n_{\mathbf{k}}} . \tag{4}$$

The factor $[\varphi_{\mathbf{k}}]^{n_k}$ in the last expression stands for the product $\varphi_{\mathbf{k}}(\mathbf{r}_{\nu+1}) \cdot \varphi_{\mathbf{k}}(\mathbf{r}_{\nu+2}) \cdot \ldots \cdot \varphi_{\mathbf{k}}(\mathbf{r}_{\nu+n_k})$. The total number of atoms is denoted by N.

As explained in the Introduction, a finite correlation energy (3) requires finite extensions of the phase ordered single particle functions. For this purpose the

4

volume V is thought to be divided into V/V_0 subvolumes of size V_0, and the single particle functions are restricted to such subvolumes. The phase ordering can then be obtained by requiring physical boundary conditions for the subvolumes, or by considering coherent states in these subvolumes (Sec. 2.3). The product over the subvolumes might be explicitly displayed in the last expression of Eq. (4). We will adjust the prefactor of the momentum integrals such that the result corresponds to the total volume V.

The Jastrow factors are of no influence on the statistical counting. Without phase ordering (i.e., using the single particle functions (1) in Eq. (4)) one obtains, therefore, the well-known IBG expression for the entropy:

$$S_{\text{IGB}} = k_{\text{B}} \sum_{\mathbf{k}} \left[(1 + n_{\mathbf{k}}) \ln(1 + n_{\mathbf{k}}) - n_{\mathbf{k}} \ln(n_{\mathbf{k}}) \right]. \tag{5}$$

The entropy change due to the phase ordering may be divided into two parts:

(a) All ν_q atoms in states with $k_1 = q$ must adopt the same (mean) phase. This leads to a replacement of the entropy (5) by a modified expression S'_{IBG}.

(b) A well-defined phase requires a small phase variance. The entropy change due to this restriction is denoted by $S_{\text{p.o.}}$. This decisive contribution is called *phase ordering entropy*.

2.2   Mean phase

Admitting the values $\phi = 0$ and $\phi = \pi/2$ in Eq. (2) means that we use the functions $\sin(qx)$ and $\cos(qx)$. This is equivalent to the use of $\exp(ik_1 x)$ and $\exp(-ik_1 x)$, i.e., to the case (1) of no phase ordering. Using the single particle functions (2) with one definite phase (lets say $\phi = 0$) effectively implies that only every second single particle state is occupied. This can be accounted for by the substitutions $\sum_{\mathbf{k}} \to (1/2) \sum_{\mathbf{k}}$ and $n_{\mathbf{k}} \to 2n_{\mathbf{k}}$ in the expression (5), i.e.,

$$S'_{\text{IBG}} = \frac{k_{\text{B}}}{2} \sum_{\mathbf{k}} \left[(1 + 2n_{\mathbf{k}}) \ln(1 + 2n_{\mathbf{k}}) - 2n_{\mathbf{k}} \ln(2n_{\mathbf{k}}) \right]. \tag{6}$$

The step from the entropy (5) to the entropy (6) takes into account the choice $\phi = 0$ for all atoms. Deviating from this, we could admit different ϕ_q values in Eq. (2) for different q's without destroying the considered correlations. This is, however, a negligible effect because the increase in entropy would be $\Delta S/N = \sum_q \ldots /N \sim \mathcal{O}(N^{-2/3}) \approx 0$.

The expectation value of S'_{IBG} has similar temperature dependence as that of S_{IBG} (Sec. 3.2).

2.3 Phase variances

Anderson [6] has established a connection between the condensate wave function and *locally coherent states*. The coherent states are constructed in suitably chosen subvolumes ΔV. The volumes must be large enough in order to define a mean phase $\bar{\phi}$ with a small variance $\Delta\phi \ll 1$; this requires that the number of atoms in ΔV is large compared to 1. On the other hand, the volumes ΔV must be small enough in order to ensure $\bar{\phi} \approx$ const. in spite of macroscopic perturbations or flows.

The connection between the condensate wave function and locally coherent states may be carried over to noncondensed phase ordered states (2) provided that $n_{\mathbf{k}} \gg 1$. This means that we may construct coherent states that correspond to the single particle functions (2). As in Anderson's work these coherent states are restricted to finite volumes. For the following calculation it is not necessary to quantify the size of these subvolumes.

Appendix A presents an explicit construction of coherent states that correspond to phase ordered single particle functions (2). Here we restrict ourselves to the presented argument based on the analogy to known work [6].

If the atoms with momentum $\mathbf{k}$ form a coherent state then their phase variance [20] reads

$$\Delta\phi_{\mathbf{k}} = \frac{1}{2\sqrt{\bar{n}_{\mathbf{k}}}} \, . \tag{7}$$

The bar denotes the quantum mechanical average.

In Eq. (1) we may admit arbitrary phases ϕ_j for each atom, $\varphi_{\mathbf{k}}(\mathbf{r}_j) \propto \exp(\mathrm{i}\,\mathbf{k} \cdot \mathbf{r}_j + \phi_j)$. These phases ϕ_j are of no influence on physical quantities (like the single particle kinetic energy or an interaction matrix element). In particular, they are of no influence on the statistical counting. Such phases are usually ignored.

Contrary to this, arbitrary phases ϕ_j in Eq. (2), $\varphi_{\mathbf{k}}^{\text{p.o.}} \propto \sin(qx_j + \phi_j)$, would destroy the considered correlations. The phase ordering requires, therefore, that these phases ϕ_j are adequately restricted, i.e., to an interval of the order (7) around a common mean value. A random phase corresponds to a phase variance $\Delta\phi_{\text{ra}} = \pi/\sqrt{3}$. (We use the phase definition by Barnett et al. [20]; the numerical factors are, however, without influence on the critical part of the entropy.) Restricting the phase ϕ_j of the jth atom to the interval (7) corresponds to a reduction factor $\Delta\phi_{\mathbf{k}}/\Delta\phi_{\text{ra}}$. For all atoms this leads to the entropy contribution

$$S_{\text{p.o.}} = k_{\text{B}} \ln \prod_{\mathbf{k}} \left(\frac{\Delta\phi_{\mathbf{k}}}{\Delta\phi_{\text{ra}}} \right)^{\bar{n}_{\mathbf{k}}} \approx -\frac{k_{\text{B}}}{2} \sum_{\mathbf{k}} \bar{n}_{\mathbf{k}} \ln(\bar{n}_{\mathbf{k}}) \qquad (\bar{n}_{\mathbf{k}} \gg 1). \tag{8}$$

This expression appears to be the most simple and plausible way of accounting for the phase restrictions.

Phase ordering requires that the considered phases are reasonably well defined, i.e., $\Delta\phi_{\mathbf{k}} \ll 1$ or $\bar{n}_{\mathbf{k}} \gg 1$. The condition $\bar{n}_{\mathbf{k}} \gg 1$ has been used in the last step in Eq. (8). We note that the critical terms in the expectation value of the entropy are solely due to the contributions from $\bar{n}_{\mathbf{k}} \gg 1$. The expectation value of $S_{\text{p.o.}}$ will exhibit a logarithmic singularity (Sec. 3.3), i.e., the phase ordering entropy is the decisive contribution in our model.

For $n_{\mathbf{k}} \gg 1$, expression (5) yields $S_{\text{IBG}} \sim \sum k_{\text{B}} \ln n_{\mathbf{k}}$. From this and Eq. (8) it follows that the low $\mathbf{k}$ contributions to the sum $S_{\text{IBG}} + S_{\text{p.o.}}$ are negative which might seem disturbing. The entropy must, however, not be attributed separately to each $\mathbf{k}$ level; it rather results from the distribution of all particles onto the available levels. Consider, for example, an isolated level with the single particle energy ε_1 that is occupied by n_1 atoms. Then we may attribute the energy $\varepsilon_1 n_1$ to this level but we cannot say that $k_{\text{B}} \ln n_1$ is the entropy of this (isolated) level. Alternatively one may argue as follows: The low $\mathbf{k}$ contributions to S_{IBG} (or S'_{IBG}) are small and noncritical. Consequently, any entropy expression that leads to the experimentally observed critical behavior (i.e., to $S_{\text{exp}} \propto -t \ln |t|$, where t is the relative temperature) must override the low $\mathbf{k}$ contribution of S_{IBG} (similarly as it is done by $S_{\text{p.o.}}$).

3 Thermodynamic entropy

3.1 Statistical assumptions

In order to determine the thermodynamic entropy we need the expectation values $\langle n_{\mathbf{k}} \rangle$ of the occupation numbers. For the wave function (4) with the single particle functions (1) and with hard sphere Jastrow factors, Chester [5] obtained the IBG values, i.e., $\langle n_{\mathbf{k}} \rangle = \langle n_{\mathbf{k}} \rangle_{\text{IBG}}$. This means that the IBG occupation numbers remain valid in spite of hard core interactions. The considered phase ordering correlations are small in the sense that $|E_{\text{corr}}|/N \ll k_{\text{B}} T_\lambda$. Therefore, it appears to be a sensible approximation to use the IBG occupation numbers for the phase ordered states, too. The phenomenological assumption

$$\langle n_{\mathbf{k}} \rangle = \langle n_{\mathbf{k}} \rangle_{\text{IBG}} \tag{9}$$

is the basis of our description of the phase transition. It implies that we consider an *almost* ideal Bose gas model. This classification refers to the statistical assumption (9) but not to an assumption about weak interactions.

The thermodynamic entropy is obtained by

$$S(T, V, N) = \langle S(n_{\mathbf{k}}) \rangle = S(\langle n_{\mathbf{k}} \rangle), \tag{10}$$

where

$$S(n_{\mathbf{k}}) = S'_{\mathrm{IBG}} + S_{\mathrm{p.o.}}. \tag{11}$$

The contributions S'_{IBG} and $S_{\mathrm{p.o.}}$ are defined in Eqs. (6) and (8), respectively. For the relevant states ($n_{\mathbf{k}} \gg 1$) the quantum mechanical variances $\Delta n_{\mathbf{k}} / \bar{n}_{\mathbf{k}} \sim 1/\sqrt{\bar{n}_{\mathbf{k}}} \ll 1$ are small compared to the statistical variances. Therefore, we omit a distinction between $\langle \bar{n}_{\mathbf{k}} \rangle$ and $\langle n_{\mathbf{k}} \rangle$.

The average occupation numbers of the IBG are

$$\langle n_{\mathbf{k}} \rangle = \frac{1}{\exp\left[(\varepsilon_{\mathbf{k}} - \mu)/k_{\mathrm{B}}T \right] - 1} = \frac{1}{\exp\left(x^2 + \tau^2 \right) - 1}. \tag{12}$$

Here $\varepsilon_{\mathbf{k}} = \hbar^2 k^2 / 2m$ is the single particle energy and μ is the chemical potential. We use the dimensionless quantities x and τ,

$$x^2 = \frac{\varepsilon_{\mathbf{k}}}{k_{\mathrm{B}}T} = \frac{\lambda^2 k^2}{4\pi} \quad \text{and} \quad \tau^2 = -\frac{\mu}{k_{\mathrm{B}}T}, \tag{13}$$

where $\lambda = 2\pi\hbar/\sqrt{2\pi m k_{\mathrm{B}}T}$ denotes the thermal wave length. The transition temperature T_λ is given by the condition

$$\lambda(T_\lambda) = \left[v\, \zeta(3/2) \right]^{1/3}, \tag{14}$$

where $\zeta(3/2) \approx 2.6124$ denotes Riemann's zeta function and $v = V/N$ is the volume per particle. In the following we use the relative temperature

$$t = t(T, v) = \frac{T - T_\lambda}{T_\lambda}. \tag{15}$$

The chemical potential μ vanishes at the transition point. For $|t| \ll 1$ the particle number condition $N = \sum \langle n_{\mathbf{k}} \rangle$ yields

$$\tau(t) = \sqrt{\frac{-\mu}{k_{\mathrm{B}}T}} = \begin{cases} a\,t + b\,t^2 + \dots & (t > 0) \\ a'\,|t| + b'\,t^2 + \dots & (t < 0). \end{cases} \tag{16}$$

The coefficients are known for the IBG, in particular $a = (3/4)\zeta(3/2)/\sqrt{\pi}$ and $a' = b' = \dots = 0$. As a generalization of the IBG we admit $\tau = a'\,|t| + \dots$ with $a' \geq 0$ for $t < 0$. This generalization does not change the character of the transition

8

(neither the point of transition nor the critical exponents). Using the second line in Eq. (16) and the particle number condition we obtain the condensate fraction

$$\frac{n_0(t)}{N} = \left(\frac{3}{2} + \frac{2\sqrt{\pi}\, a'}{\zeta(3/2)}\right)|t| + g\, t^2 + \dots . \tag{17}$$

Here $n_0(t)$ is the expectation value of the occupation number of the lowest energy level.

There are a number of reasons for generalizing the IBG by admitting $\tau = a'\,|t| + \dots$ with a coefficient $a' \geq 0$ in the expansion (16): Formally, this generalization leads to a more symmetric form of this expansion. Physically, a positive value of a' implies a temperature dependent energy gap $\Delta \approx k_{\mathrm{B}} T a'^2 t^2$ between the condensate and the noncondensed particles and implies that the lowest level is more rapidly occupied than in the IBG. Such a behavior leads to a better agreement between the calculated and the experimental temperature dependence of various quantities, in particular of the superfluid density [21]. As a last argument we mention that $a' > 0$ removes [22] the unphysical divergence ($1/k^2$ for $k \to 0$) of the static structure function of the IBG for $t < 0$. Our main result will not depend on the parameter a'.

The well-known expectation value of the IBG entropy (5) is given by

$$\frac{S_{\mathrm{IBG}}(T, V, N)}{N k_{\mathrm{B}}} = \frac{5}{2} \frac{v}{\lambda^3}\, g_{5/2}(\tau) + \tau^2, \tag{18}$$

where

$$g_\nu(\tau) = \sum_{n=1}^{\infty} \frac{\exp(-n\tau^2)}{n^\nu} = \sum_{n=1}^{\infty} \frac{z^n}{n^\nu} \tag{19}$$

defines Riemann's generalized zeta function. Usually $z = \exp(\beta\mu) = \exp(-\tau^2)$ is taken as the argument of this function. We prefer the argument τ because of its close relation ($\tau \sim |t|$) to the relative temperature.

For the IBG entropy one has to use $\tau \equiv 0$ in Eq. (18) for $t < 0$. The modification $\tau = a'|t|$ with $a' > 0$ leads to a specific heat that is continuous at T_λ (as in the pure IBG) but that falls off more rapidly for $t < 0$.

3.2   Mean phase contribution

Because of the common mean phases the IBG expression (5) is replaced by S'_{IBG}, Eq. (6). The expectation value of the entropy S'_{IBG} is

$$\frac{S'_{\mathrm{IBG}}(T, V, N)}{N k_{\mathrm{B}}} = \frac{1}{2} \sum_{\mathbf{k}} \Big[(1 + 2\langle n_{\mathbf{k}}\rangle)\ln(1 + 2\langle n_{\mathbf{k}}\rangle) - 2\langle n_{\mathbf{k}}\rangle \ln(2\langle n_{\mathbf{k}}\rangle)\Big]. \tag{20}$$

For a first, crude estimate we take into account only those contributions that come from $\langle n_{\mathbf{k}} \rangle \gg 1$. This yields $S_{\mathrm{IBG}} \sim k_{\mathrm{B}} \sum_{\mathbf{k}} \ln \langle n_{\mathbf{k}} \rangle$ and $S'_{\mathrm{IBG}} \sim (k_{\mathrm{B}}/2) \sum_{\mathbf{k}} \ln \langle n_{\mathbf{k}} \rangle$, i.e., $S'_{\mathrm{IBG}}(T, V, N) \sim S_{\mathrm{IBG}}(T, V, N)/2$.

The actual results for S_{IBG} and S'_{IBG} near the transition point are

$$S_{\mathrm{IBG}}(T, V, N) \approx 1.28 \, N k_{\mathrm{B}} \left(1 + \frac{3}{2}t + \mathcal{O}(t^2)\right), \tag{21}$$

$$S'_{\mathrm{IBG}}(T, V, N) \approx 0.96 \, N k_{\mathrm{B}} \left(1 + \frac{3}{2}t + \mathcal{O}(t^2)\right). \tag{22}$$

Eq. (21) follows from the expression (18). The value of S'_{IBG} at the lambda point has been determined by a numerical integration of the r.h.s. of Eq. (20). The next term in the expansion is due to the prefactor $v/\lambda^3 \propto T^{3/2}$ in Eq. (18). This prefactor originates from the replacement of the momentum sum by an integral over dimensionless variables; it occurs in both entropy expressions in the same way. The higher terms $\mathcal{O}(t^2)$ in Eqs. (21) and (22) do not agree; moreover, the coefficients of these t^2 terms are different for $t > 0$ and $t < 0$.

3.3 Phase variance contribution

The expectation value of the phase ordering entropy $S_{\mathrm{p.o.}}$, Eq. (8), reads

$$\frac{S_{\mathrm{p.o.}}(T, V, N)}{N k_{\mathrm{B}}} = -\frac{1}{2N} \sum_{\mathbf{k}} \langle n_{\mathbf{k}} \rangle \, \ln \langle n_{\mathbf{k}} \rangle \qquad (|t| < 0.1). \tag{23}$$

For the phase ordering we required $\Delta \phi_{\mathbf{k}} \ll 1$ in Eq. (8), lets say $\Delta \phi_{\mathbf{k}} < 0.1$ or $\langle n_{\mathbf{k}} \rangle > 10^2$. Because of $\langle n_{\mathbf{k}} \rangle \approx 1/(x^2 + \tau^2)$ this implies $x \lesssim 0.1$ and $\tau \lesssim 0.1$ or $|t| < 0.1$. The condition $|t| < 0.1$ indicates the temperature range in which phase ordering and the corresponding entropy contribution are expected to be relevant.

For evaluating the entropy (23) the momentum sum is replaced by an integral,

$$\sum_{\mathbf{k}} \dots \longrightarrow \frac{V}{(2\pi)^3} \int_0^\infty dk \, 4\pi k^2 \dots . \tag{24}$$

This step has to be accompanied by a separate consideration of a potential condensate fraction; this will be done below in Eq. (28).

For a subvolume of size V_0 the r.h.s. of Eq. (24) would be $V_0/(2\pi)^3 \int_0^\infty dk \, 4\pi k^2$. The subsequent summation over all subvolumes yields the factor V/V_0; this summation has been taken into account in the prefactor on the r.h.s. of Eq. (24). In

Appendix B we discuss the validity of the step (24) with respect to finite spacing $\Delta k = \pi/V_0^{1/3}$ of the momentum values.

We insert the replacement (24), the dimensionless quantities (13) and the occupation numbers (12) into Eq. (23):

$$\frac{S_{\text{p.o.}}}{Nk_{\text{B}}} = \frac{2}{\sqrt{\pi}} \frac{v}{\lambda^3} \int_0^\infty dx\, x^2\, \frac{\ln[\exp(x^2 + \tau^2) - 1]}{\exp(x^2 + \tau^2) - 1} = \frac{2}{\sqrt{\pi}} \frac{v}{\lambda^3}\, J(\tau). \qquad (25)$$

The critical τ dependence of the integral $J(\tau)$ can be determined analytically (App. C),

$$J(\tau) = J(0) - \pi\tau \ln\tau + \pi\tau[1 - \ln(2)] + \mathcal{O}(\tau^2). \qquad (26)$$

Using this, $\tau = (3\,\zeta(3/2)/4\sqrt{\pi}\,)\,|t| + \mathcal{O}(t^2)$, and $v/\lambda^3 = 1/\zeta(3/2) + \mathcal{O}(t)$ we obtain

$$S_{\text{p.o.}}(T, V, N) = -\frac{3Nk_{\text{B}}}{2}\, t \ln|t| \pm \ldots \qquad (|t| < 0.1) \qquad (27)$$

for the leading singularity of the phase ordering entropy for $t > 0$. As we will see, this result holds for $t < 0$, too.

For $t < 0$ the prescription (24) has to be accompanied by the replacement

$$\langle n_{\mathbf{k}} \rangle \to \langle n_{\mathbf{k}} \rangle + n_0(t)\, \delta(\mathbf{k}). \qquad (28)$$

A phase variance $\Delta\phi_0 \sim 1/\sqrt{n_0} \approx 0$ stands for a macroscopic phase coherence. This is appropriate for a superfluid flow but not for the considered local phase ordering. For the phase variances (7) we use, therefore, the continuous part of the occupation numbers $\langle n_{\mathbf{k}} \rangle$ only. The contribution $S_{\text{p.o.}}^{\text{cond}}$ in Eq. (23) due to the condensed particles reads then

$$\frac{S_{\text{p.o.}}^{\text{cond}}}{Nk_{\text{B}}} = -\frac{n_0(t)}{2N}\, \ln\langle n_{\mathbf{k} \to 0} \rangle = \frac{n_0(t)}{2N}\, \ln[\exp(\tau^2) - 1]\,. \qquad (29)$$

This can be evaluated by using Eq. (17) for $n_0(t)/N$ and Eq. (16) for $\tau(t)$.

For $t < 0$ the contributions (25) and (29) have to be added. The a' term of $n_0(t)/N = [3/2 + 2\sqrt{\pi}a'/\zeta(3/3)]|t| + \ldots$ in Eq. (29) cancels exactly the logarithmic term in Eq. (25). The surviving leading term comes from $n_0(t)/N = (3/2)|t| + \ldots$ and $\ln[\exp(\tau^2) - 1] = 2\ln|t| + \ldots$ yielding again Eq. (27).

3.4 Specific heat

The specific heat per particle may be written as

$$c_V = \frac{1}{N}\left(\frac{\partial S}{\partial t}\right)_{V,N} = \begin{cases} -A\ln|t| + B + \ldots & (t > 0) \\ -A'\ln|t| + B' + \ldots & (t < 0)\,. \end{cases} \qquad (30)$$

From Eq. (27) we obtain the theoretical amplitudes

$$A_{\text{theor}} = A'_{\text{theor}} = \frac{3}{2} \, k_{\text{B}}. \tag{31}$$

If the experimental data for c_V are fitted by the form (30) one finds $A_{\text{exp}} \approx 0.63 \, k_{\text{B}}$ and $A'_{\text{exp}} \approx 0.59 \, k_{\text{B}}$; these values are taken from Eq. (4.2) (for low pressure) of Ref. [23]. It is remarkable that we obtain a parameter free result and a sensible size for these amplitudes. The experimental ratio $A_{\text{exp}}/A'_{\text{exp}}$ deviates from 1 by a few percent. In our model, the contribution (25) of the noncondensed particles to $S_{\text{p.o.}}$ has different signs above and below the lambda point. A correction in this contribution could, therefore, lead to a deviation from $A_{\text{theor}}/A'_{\text{theor}} = 1$.

Experimentally the specific heat c_P at constant pressure is more readily accessible. It is this quantity that has been measured [24, 25, 26, 27] over several decades of the relative temperature, eventually down to $|t| = 2 \times 10^{-9}$ in a microgravity experiment [27]. In the context of these measurements and their analyses we note the following points:

1. Theoretically, a logarithmic behavior of one of the two quantities, c_P or c_V, implies [28, 29] an analogous behavior of the other quantity in the experimentally accessible range. The statement [30] that only one of these quantities may diverge for $|t| \to 0$ applies to very low (experimentally inaccessible) $|t|$ values only [29].

   The experimental values for c_P may be fitted by the form (30), too. The corresponding coefficients (as given by Eq. (47) of Ref. [24]) differ from that for c_V by about 5 to 10%. For comparing our result (30) with the experiment we may, therefore, also consider the experimental c_P.

2. If the critical exponents α and α' are used as fit parameters one finds [24, 26, 27] small negative values. The following points indicate that this does not rule out a logarithmic singularity (corresponding to $\alpha \to 0$ and $\alpha' \to 0$) for the leading term:

   (a) Arp [31] fitted the data of many experiments in order to obtain an equation of state for helium. He considered specifically the question of the critical exponent α (Sec. 13.1.1 of Ref. [31]) and found no statistical significance for a deviation from a leading logarithmic form. Consequently, he adopted the value $\alpha = \alpha' = 0$ (standing for a logarithmic function) in his expression for the specific heat c_V.

12

(b) By the scaling law $\alpha' = 2 - 3\nu$ the critical exponent for the specific heat for $t < 0$ is related to that of the superfluid density $\varrho_s \sim |t|^\nu$. The deviations from $\alpha' = 0$ and $\nu = 2/3$ found in standard fits are consistent [27] with this scaling law.

In Ref. [21] we presented an alternative fit formula for the superfluid density in which the leading exponent ν equals exactly 2/3. This fit formula reproduced the data better than standard fit formulas. This means that the value $\nu = 2/3$ (corresponding to $\alpha' = 0$) is not ruled out by the experiment, and that the value of the leading exponent (found in a fit) might depend on the higher order terms (used in the fit).

In view of these points, we consider it to be an open question whether the true behavior deviates from a logarithmic one in a way that cannot be accounted for by higher order terms.

We turn now to the coefficients B and B' of the specific heat (30). From S'_{IBG}, Eq. (22), we get a contribution of $1.44\,k_B$ for both, B_{theor} and B'_{theor}. In Eq. (27) we add the terms that are linear in t; these terms follow from Eqs. (25) and (29). Including all contributions we obtain

$$B_{\mathrm{theor}} \approx -0.52\,k_B, \qquad B'_{\mathrm{theor}} = -3.7\,k_B. \tag{32}$$

For B'_{theor} we used the parameter value $a' \approx 3$ that is (rather uniquely) determined [21] by adjusting the model expression for the superfluid density to the experimental temperature dependence.

The value for B_{theor} compares reasonably well with the experimental value $B_{\mathrm{exp}} \approx -0.84\,k_B$ (from Eq. (4.5) of Ref. [23] for low pressure).

There is a large discrepancy between B'_{theor} and the experimental value [23] $B'_{\mathrm{exp}} \approx 2.0\,k_B$. In the IBG, the coefficient of the t^2 term in the free energy F_{IBG} is the same above and below the transition. This means that the IBG does not reproduce the jump of the specific heat that normally accompanies the occurrence of a macroscopic order parameter field. For a crude estimate of this missing jump we consider the Landau free energy $F_{\mathrm{Landau}}/(Nk_B T_\lambda) = rt|\psi|^2 + u|\psi|^4$. This Landau model yields a jump of the specific heat, $\Delta c_V/k_B = r^2/2u$. The identification $\langle|\psi|^2\rangle = n_0(t)/N$ relates the coefficient in $\langle|\psi|^2\rangle = (r/2u)|t|$ to that in Eq. (17). This connection leads to $\Delta c_V = r\,[3/2 + 2\sqrt{\pi}a'/\zeta(3/2)]$. Using again $a' \approx 3$ yields $\Delta c_V \approx 5.6\,r$. For the parameter r in the expression for $F_{\mathrm{Landau}}/(Nk_B T_\lambda)$ we may expect $r \sim \mathcal{O}(1)$. The resulting $\Delta c_V \sim 5.6\,k_B$ could close the gap between B'_{theor} of Eq. (32) and B'_{exp}.

3.5 Entropy at the lambda point

As a last point we consider the value of the entropy at the lambda point. Evaluating the phase ordering entropy (25) as it stands yields

$$S_{\text{p.o.}}(T_\lambda) = \frac{2J(0)}{\sqrt{\pi}\,\zeta(3/2)}\, Nk_{\text{B}} \approx -0.51\, Nk_{\text{B}}. \tag{33}$$

This result contains, however, contributions from momenta for which the condition $\langle n_{\mathbf{k}} \rangle \gg 1$ is not satisfied.

We estimate the value $S_{\text{p.o.}}^{\text{est}}(T_\lambda)$ that comes from the contributions with $\langle n_{\mathbf{k}} \rangle \gg 1$ alone. For small and positive t values we may assume the form $S_{\text{p.o.}}/N \approx S_{\text{p.o.}}^{\text{est}}(T_\lambda)/N - A_{\text{theor}}\, t \ln(t) + (B_{\text{p.o.}} + A_{\text{theor}})\, t$. Following Eq. (23) we argued that the phase ordering entropy should be relevant in the range $|t| < 0.1$ only. This implies $S_{\text{p.o.}}(t = 0.1) \approx 0$ leading to $S_{\text{p.o.}}^{\text{est}}(T_\lambda) \approx -0.3\, Nk_{\text{B}}$.

We conclude that the result (33) might be too large by roughly a factor of 2. One may, therefore, expect that the contribution $S_{\text{p.o.}}(T_\lambda)$ accounts for the difference between $S'_{\text{IBG}} \approx 0.96\, Nk_{\text{B}}$ and the experimental value [32] $S_{\text{exp}}(T_\lambda) \approx 0.76\, Nk_{\text{B}}$.

4 Concluding remarks

We summarize the main features of the presented phenomenological many-body model:

1. Following Chester [5] we use the IBG many-body states multiplied by Jastrow factors.

2. We assume that local phase correlations are the relevant correlations near the lambda point. These correlations are specified by the single particle functions (2).

3. The local phase ordering can be described by coherent states. The phase variances of these coherent states are the basis for the statistical counting of the phase restrictions. This leads to the expression (8) for the phase ordering entropy.

4. For evaluating the phase ordering entropy we use the IBG expression for the average occupation numbers.

It is clear that global phase ordering plays a decisive role in liquid helium below the transition point. This makes our assumption that local phase ordering are relevant correlations near the lambda point to some extent plausible. We have proposed a specific kind of the local phase ordering in the framework of an almost ideal Bose gas model. The use of the IBG occupation numbers is a phenomenological assumption.

Our model yields a logarithmic singularity of the specific heat in a straightforward and rather simple way: The expression (8) for the phase ordering entropy follows from the phase variances (7). Evaluating this expression with the IBG occupation number yields the logarithmic singularity (27) (immediately for $t > 0$, and after a slight generalization of the occupation numbers for $t < 0$).

We summarize the main results of our almost ideal Bose gas model:

1. The model yields a realistic expression for the specific heat of liquid helium at the λ transition. For the strength of the logarithmic singularity the remarkable result $A = A' = 3\,k_{\mathrm{B}}/2$ is obtained.

2. The model retains the essential features of the IBG. Therefore, it strengthens the suggested relation [1, 2] between the Bose-Einstein condensation and the λ transition.

3. The model offers an intriguing picture for the relevant correlations near the λ point. This picture provides also perspectives beyond the specific heat (some of which are indicated in App. B).

Various other quantities may be calculated in the framework of our almost ideal Bose gas model: In an earlier paper [4] we calculated the energy as a function of the temperature; the present derivation of the specific heat appears to be more direct. The critical exponent $\beta = 1/2$ of the condensate fraction may be reconciled with the actual behavior of the superfluid density by assuming that noncondensed particles move coherently with the condensate [21]. This idea leads to observable consequences that have been discussed in Ref. [33].

A Coherent states

In order to justify the use of relation (7) we establish a connection between the phase ordered single particle functions (2) and coherent states. Our procedure is analogous to Anderson's construction [6] of coherent states for the condensate.

The many-body wave function (4) can be written in the form

$$\Psi = \mathcal{S}F \prod_{j=1}^{n_0} \varphi_0(\mathbf{r}_j) \prod_{j=n_0+1}^{n_0+n_1} \varphi_1(\mathbf{r}_j) \cdot \ldots . \tag{34}$$

The single particle functions (phase ordered or not) are denoted by φ_i, where $i = 0, 1, 2, \ldots$ follows the energy sequence. Without the Jastrow factor the off-diagonal single particle density reads

$$\varrho(\mathbf{r}, \mathbf{r}') = \left\langle \Psi \left| \hat{\psi}^+(\mathbf{r}') \, \hat{\psi}(\mathbf{r}) \right| \Psi \right\rangle = n_0 \, \varphi_0(\mathbf{r})^* \varphi_0(\mathbf{r}') + n_1 \, \varphi_1(\mathbf{r})^* \varphi_1(\mathbf{r}) + \ldots , \tag{35}$$

where $\hat{\psi}^+$ and $\hat{\psi}$ are the particle creation and annihilation operator, respectively. Including the Jastrow factor the condensate contribution in Eq. (35) is depleted [34, 35] from n_0 to a lower value n_c. Similar effects are to be expected for next low-momentum contributions.

All terms in Eq. (35) besides the first term vanish for $|\mathbf{r} - \mathbf{r}'| \rightarrow \infty$. This *off-diagonal long range order* [34] is considered [36] as the decisive criterium for superfluidity.

Anderson [6] related the off-diagonal long range order to the mean field aspect by constructing localized coherent states. The coherent states $|\text{coh}\rangle$ are defined such that $\langle\text{coh}|\hat{\psi}|\text{coh}\rangle \propto \exp(\mathrm{i}\phi)$, where ϕ is the phase of the condensate wave function $\varphi_0 \propto \exp(\mathrm{i}\phi)$. Accordingly, we will construct coherent states $|\text{coh}\rangle$ for which $\langle\text{coh}|\hat{\psi}|\text{coh}\rangle$ is proportional to the phase ordered single particle function (2). Anderson's aim was to show that the phase of the condensate and the corresponding particle number may be treated as macroscopic variables. Our aim is to justify Eq. (7) by relating the wave functions (2) to coherent states.

Anderson [6] introduced finite volumes ΔV in which the phase ϕ of the condensate wave function φ_0 is approximately constant. We consider phase ordered single particle functions that are restricted to finite subvolumes of size V_0. In the following we identify ΔV with V_0 and treat both cases simultaneously.

The single particle functions φ_k shall form an orthonormal set in one subvolume of size V_0. Then we may write down the following relations between the field

operator $\hat{\psi}(\mathbf{r})$ and the annihilation operator $\hat{c}_{\mathbf{k}}$:

$$\hat{\psi}(\mathbf{r}) = \sum_{\mathbf{k}} \varphi_{\mathbf{k}}(\mathbf{r})\, \hat{c}_{\mathbf{k}}\,, \tag{36}$$

$$\hat{c}_{\mathbf{k}} = \int_{V_0} d^3r\, \varphi_{\mathbf{k}}^*(\mathbf{r})\, \hat{\psi}(\mathbf{r}). \tag{37}$$

A state with $n_{\mathbf{k}}$ particles of momentum $\mathbf{k}$ is given by

$$|n\rangle = |n_{\mathbf{k}}\rangle = \frac{1}{\sqrt{n_{\mathbf{k}}!}}\, \hat{c}_{\mathbf{k}}^+\, |\text{vac}\rangle, \tag{38}$$

where $|\text{vac}\rangle$ is the vacuum state. In the following we restrict ourselves to one specific momentum and omit the index $\mathbf{k}$. The coherent state [37] is constructed as

$$|\text{coh}\rangle = \exp(-|z|^2/2) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\, |n\rangle. \tag{39}$$

This state depends on a complex number z. Because of $\bar{n} = \langle\text{coh}|\hat{c}_{\mathbf{k}}^+\hat{c}_{\mathbf{k}}|\text{coh}\rangle = |z|^2$ we may set

$$z = \sqrt{\bar{n}}\, \exp(\mathrm{i}\phi). \tag{40}$$

For the state (39) with Eqs. (38) and (40) we find

$$\langle\text{coh}|\hat{\psi}(\mathbf{r})|\text{coh}\rangle = \sqrt{\bar{n}}\, \exp(\mathrm{i}\phi)\, \varphi_{\mathbf{k}}(\mathbf{r}) \tag{41}$$

for the expectation value of the field operator.

Using

$$\varphi_0(\mathbf{r}) = \begin{cases} 1/\sqrt{V_0} & (\mathbf{r} \in V_0) \\ 0 & (\mathbf{r} \notin V_0)\,, \end{cases} \tag{42}$$

for the lowest state, Eq. (41) becomes $\langle\text{coh}|\hat{\psi}|\text{coh}\rangle = \sqrt{\bar{n}/V_0}\, \exp(\mathrm{i}\phi)$. So far we have reproduced Anderson's construction.

We are now going to connect the phase ordered single particle functions (2) with coherent states. For this purpose we introduce the single particle functions

$$\varphi_{\mathbf{k},+}(\mathbf{r}) \quad \propto \quad \exp\left[\mathrm{i}\left(k_2 y + k_3 z\right)\right] \exp(+\mathrm{i}qx) \tag{43}$$

$$\varphi_{\mathbf{k},-}(\mathbf{r}) \quad \propto \quad \exp\left[\mathrm{i}\left(k_2 y + k_3 z\right)\right] \exp(-\mathrm{i}qx) \tag{44}$$

These functions shall be orthonormalized in the considered subvolume. Analogously to Eqs. (38) with (37) we introduce the n-particle states $|n\rangle_+$ and $|n\rangle_-$ that

correspond to $\varphi_{\mathbf{k},+}$ and $\varphi_{\mathbf{k},-}$, respectively. The coherent state

$$|\mathrm{coh}\rangle = \frac{\exp(-|z|^2/2)}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{z^n |n\rangle_+ - z^{*n} |n\rangle_-}{\sqrt{n!}} \tag{45}$$

with $z = \sqrt{\bar{n}}\exp(\mathrm{i}\phi)$ yields then

$$\langle\mathrm{coh}|\hat{\psi}(\mathbf{r})|\mathrm{coh}\rangle = \sqrt{\bar{n}}\,\varphi_{\mathbf{k}}^{\mathrm{p.o.}}(\mathbf{r}). \tag{46}$$

The phase variance of the coherent state (45) is given by

$$\Delta\phi_{\mathbf{k}} = \frac{1}{2\sqrt{\bar{n}_{\mathbf{k}}}}. \tag{47}$$

Here we have attached the index $\mathbf{k}$ again; the above discussion referred to a single $\mathbf{k}$ value. Eq. (47) is a well-known result [20] for a coherent state of the standard form (39). For the state (45) we may write $|\mathrm{coh}\rangle = (|\mathrm{coh}\rangle_+ - |\mathrm{coh}\rangle_-)/\sqrt{2}$ in an obvious notation. The phase operator acts separately within each of the two sets of states, $\{|n\rangle_+\}$ or $\{|n\rangle_-\}$. Moreover, the states of one of these sets are orthogonal to that of the other set (because of the functions (43) and (44) are mutually orthogonal). Therefore, the phase variance of the state (45) is one half of the sum of the phase variances of the states $|\mathrm{coh}\rangle_+$ and $|\mathrm{coh}\rangle_-$ that are of the standard form (39). This means that we obtain the standard result (47) for our somewhat special coherent state (45), too.

The reason for establishing the relation between the phase ordered single particle functions (2) and coherent states (45) is the necessity to justify the relation (47) or, equivalently, Eq. (7). The relation (7) is the basis for the phase ordering entropy (8).

Instead of $|\mathrm{coh}\rangle = (|\mathrm{coh}\rangle_+ - |\mathrm{coh}\rangle_-)/\sqrt{2}$ we may also consider the coherent state $(|\mathrm{coh}\rangle_+ + |\mathrm{coh}\rangle_-)/\sqrt{2}$. This orthogonal state corresponds to a cosinus in Eq. (2) instead of a sinus. The considered correlation effect requires that only one of these sets of states is occupied. This $2:1$ reduction has been discussed in Section 2.2.

With respect to the condition $n_{\mathbf{k}} \gg 1$ we note that the average occupation numbers $\langle n_{\mathbf{k}}\rangle$, Eq. (12), do not depend on the size V_0 of the subvolumes. The finite size implies, however, a finite spacing of the momentum values (App. B) that takes care of the finite number $N_0 = V_0/v$ of atoms in one subvolume.

B   Phase coherence volume

We start by explaining why finite phase correlations require that the single particle functions (2) have to be localized within finite volumes. Then we show that the lowest single particle functions may extend over several of these subvolumes. This leads to the notion of a phase coherence volume for which we obtain $V_{\text{coh}} \sim V_0/|t|^2$, where V_0 is the size of one subvolume. We discuss this result in a number of points.

Two phase ordered states (2) with momentum $\mathbf{k}$ and $\mathbf{k}'$ are correlated if $k_1 = k_1'$. The restriction $k_1 = k_1'$ cancels one momentum sum in the expression for the energy. A momentum sum $\sum_k \to \int dk/\Delta k$ is proportional to $1/\Delta k = V^{1/3}/2\pi$ or to $N^{1/3}$. The correlation energy E_{corr} is, therefore, proportional to $N^{2/3}$ only. This is consistent with the observation that a phase ordering of the kind (2) can be obtained by physical boundary conditions at the wall of the considered volume. For an infinite volume such a surface effect vanishes ($E_{\text{corr}}/N \propto 1/N^{1/3} \to 0$). A finite correlation effect may, however, be obtained by dividing the total volume V into V/V_0 subvolumes of size V_0 and by requiring physical boundary conditions at the walls of these subvolumes. For the whole system (including a factor V/V_0 for summing over all subvolumes) we obtain then a result of the form

$$\frac{E_{\text{corr}}}{N} = \frac{w_0}{N_0^{1/3}} \, , \tag{48}$$

where $w_0 < 0$ is some average strength of the attractive part of the interaction (for a more detailed discussion see Ref. [4]). This result shows that a finite correlation effect requires a finite size $V_0 = N_0\,v$ of the subvolumes.

Finite volumes V_0 imply a finite spacing $\Delta k = \pi/V_0^{1/3}$ of the momentum values. Alternatively one may start from a finite spacing and admit only the momentum values

$$q_n = q_0 + n \cdot \Delta k, \qquad (n = 0, 1, 2, \ldots), \tag{49}$$

in the single particle functions (2). The spacing (49) leads to a finite correlation energy even if the single particle functions are not localized. In a macroscopic system the possible q values are, however, dense, and the entropy drives the particles into the occupation of all available states. This is the reason why the finite spacing (49) can be realized only for single particle functions localized within subvolumes of the size $V_0 = (\pi/\Delta k)^3$. In this sense, a finite spacing implies finite volumes.

This discussion shows that we may either start from finite volumes or from a finite Δk value. There is, however, the following difference: Starting from the finite spacing (49) we may admit a q_0 value *below* Δk without damaging the correlation

energy. A single particle function with $q_0 < \Delta k$ can, however, not be localized within V_0.

In view of this observation we consider the following modified picture. Only the single particle functions (2) with $q_{n \geq 1}$ are localized within V_0, the single particle function φ_0 with q_0 may have a larger extension. We present an estimate for the volume V_{coh} of the lowest single particle function φ_0: Let V_{coh} be some multiple of V_0, that means $V_{\mathrm{coh}} = W V_0$. In this case, there are W single particle states with $q < \Delta k$ in the volume V_{coh} out of which only one (the one with q_0) is occupied. A redistribution of n_0 atoms over these W states would increase the entropy by $\Delta s \sim k_{\mathrm{B}} \ln(n_0)^W$. At the same time these atoms would loose their correlation energy, $\Delta e_{\mathrm{corr}} \sim -n_0 w_0 / N_0^{1/3}$. The stability condition $T_\lambda \Delta s < \Delta e_{\mathrm{corr}}$ yields the upper bound $W < [-w_0/(N_0^{1/3} k_{\mathrm{B}} T_\lambda)] \, n_0 / \ln n_0 \sim n_0$. Using $n_0 = \langle n_{\mathbf{k} \to 0} \rangle = \tau^{-2} \sim t^{-2}$ leads to

$$V_{\mathrm{coh}}(t) = W \, V_0 \sim \frac{V_0}{t^2} \,. \tag{50}$$

We discuss this result in a number of points:

1. The lowest single particle functions φ_0 extend over volumes of the size V_{coh}. Within a volume V_{coh} the function φ_0 defines the direction of the phase ordering (x direction in Eq. (2)) and the phase ϕ. Therefore, $V_{\mathrm{coh}}(t)$ of Eq. (50) constitutes a *phase coherence volume*.

2. Strictly finite volumes for all particles would mean that the lower bound of the integral $J(\tau)$ in Eq. (25) would be Δx instead of zero. This would imply a cut of the logarithmic singularity. The replacement (24) may, however, be justified by a lower bound $q_0 \sim 1/V_{\mathrm{coh}}$ and an average over somewhat different q_0 and Δk values in different parts of the macroscopic system. A closer examination of this point might lead to a modification of the logarithmic singularity at very low $|t|$ values.

3. In the estimate leading to Eq. (50) we used the continuous part of the occupation number $\langle n_{\mathbf{k} \to 0} \rangle$ also for $t < 0$. Using instead the condensate occupation number $n_0(t)$ leads to an infinite volume, $V'_{\mathrm{coh}} = \infty$. This is the adequate phase coherence volume for the potentially macroscopic range of a superfluid flow.

4. Different phase coherence volumes of size V_{coh} within the macroscopic system will, in general, correspond to different phase directions (x direction in Eq. (2)). A specific phase direction implies a small anisotropy of the static

structure function. In principle, the volumes V_{coh} and their increasing size for $t \to 0$ should, therefore, be observable.

5. When approaching the lambda point from above the phase coherence volumes grow according to Eq. (50). At the lambda point the coherence volume becomes infinite. This picture has some similarity with a ferromagnetic system.

   Below the lambda point the finite volume (50) refers to the phase ordering of real single particle functions of the form (2). At the same time there is a potentially infinite range of phase coherence of a superfluid current. The corresponding phase coherence volume $V'_{\text{coh}} = \infty$ refers to the complex condensate wave $\varphi_0 \propto \exp[\,i\,\phi(\mathbf{r})]$.

6. Adjusting the correlation energy (48) to the experimental strength of the logarithmic singularity yields [4] $N_0^{1/3} \approx 50$. The phase ϕ should be approximately constant within V_0 (see App. A). It appears, therefore, tempting (but also speculative) to set $|\nabla\phi|_{\text{max}} \sim V_0^{-1/3}$ and to use the relation $\mathbf{u}_{\text{s}} = \hbar\,\nabla\phi/m$ for the velocity of a potential superfluid flow. The maximum (or critical) velocity for such a flow would then be

$$v_{\text{crit}} = \frac{\hbar\,|\nabla\phi|_{\text{max}}}{m} \sim \frac{\hbar}{m\,V_0^{1/3}} = \frac{\hbar}{m\,v^{1/3}}\frac{1}{N_0^{1/3}} \approx 0.8\,\frac{\text{m}}{\text{s}}\,. \tag{51}$$

In this way the finite size V_0 renders a possible connection between the "natural" velocity scale $\hbar/(m\,v^{1/3})$ and realistic values for the critical velocity. Using Eq. (50) we obtain $v_{\text{crit}} \sim |t|^{-2/3}\,\text{m/s}$ for the critical velocity near the transition point.

C Evaluation of $J(\tau)$

We determine the critical $(\tau \ll 1)$ behavior of the integral

$$J(\tau) = \int_0^\infty dx\, x^2\, \frac{\ln[\exp(x^2 + \tau^2) - 1]}{\exp(x^2 + \tau^2) - 1}\,. \tag{52}$$

Writing $\ln[\exp(x^2 + \tau^2) - 1] = \ln[1 - \exp(-x^2 - \tau^2)] + x^2 + \tau^2$ we obtain

$$J(\tau) = \int_0^\infty dx\, \frac{x^2\,\ln[1 - \exp(-x^2 - \tau^2)]}{\exp(x^2 + \tau^2) - 1} + \int_0^\infty dx\, \frac{x^2\,(x^2 + \tau^2)}{\exp(x^2 + \tau^2) - 1}\,. \tag{53}$$

The second integral yields a result that is of the form

$$R_0 = \text{const.} + \mathcal{O}(\tau^2). \tag{54}$$

In the following R_0 stands for any expression of this kind. By $\mathcal{O}(\tau^2)$ we mean terms that are proportional to τ^2 or to higher powers of τ.

In the first integral in Eq. (53) we use the following expansions into powers of $y = x^2 + \tau^2$ or $\exp(-y)$:

$$\frac{\ln[1 - \exp(-y)]}{\exp(y) - 1} = \begin{cases} \dfrac{\ln(y)}{y} - \dfrac{\ln(y) + 1}{2} + \dfrac{y\,(2\ln(y) + 7)}{24} \pm \ldots & (y \le 1) \\[3mm] -\exp(-2y) - \dfrac{3\exp(-3y)}{2} - \dfrac{11\exp(-4y)}{6} \pm \ldots & (y \ge 1). \end{cases} \tag{55}$$

We divide the integration into one part from 0 to 1 and another part from 1 to ∞, and insert the appropriate expansion (55). All terms except the one with $\ln(y)/y$ yield contributions of the form (54). We evaluate the remaining integral over $x^2 \ln(y)/y = \ln(y) - \tau^2 \ln(y)/y$,

$$\begin{aligned} J(\tau) &= \int_0^1 dx\, \ln(x^2 + \tau^2) - \tau^2 \int_0^1 dx\, \frac{\ln(x^2 + \tau^2)}{x^2 + \tau^2} + R_0 \\ &= \pi\tau - \tau^2 \int_0^\infty dx\, \frac{\ln(x^2 + \tau^2)}{x^2 + \tau^2} + R_0. \end{aligned} \tag{56}$$

The first integral yielded $\pi\tau + R_0$ (see number 2733 in Ref. [38]). Because of $\int_1^\infty dx\, \ln(x^2 + \tau^2)/(x^2 + \tau^2) = R_0$ the upper limit of the remaining integral could be set equal to infinity. In this integral we substitute $x = \tau z$:

$$\begin{aligned} J(\tau) &= \pi\tau - \tau \int_0^\infty dz\, \frac{\ln(\tau^2)}{1 + z^2} - \tau \int_0^\infty dz\, \frac{\ln(1 + z^2)}{1 + z^2} + R_0 \\ &= J(0) - \pi\tau \ln\tau + \pi\tau[1 - \ln(2)] + \mathcal{O}(\tau^2). \end{aligned} \tag{57}$$

The first integral is elementary, the second one may be found under number 4.295 in Ref. [38]. A numerical integration yields $J(0) \approx -1.183$.

References

[1] F. London, *Superfluids* (Wiley, New York 1954), Vol. II.

[2] R. P. Feynman, Phys. Rev. **91**, 1291 (1953).

[3] S. J. Putterman, *Superfluid Hydrodynamics* (North Holland, London 1974).

[4] T. Fliessbach, Nuovo Cimento D **13**, 211 (1991).

[5] G. V. Chester, Phys. Rev. **100**, 455 (1955).

[6] P. W. Anderson, Rev. Mod. Phys. **38**, 298 (1966).

[7] J. Navarro J and A. Polls (eds.), *Microscopic Quantum Many-Body Theories and their Applications*, (Proceedings, Valencia, Spain 1997), Springer-Verlag, Berlin 1998.

[8] A. D. Jackson, A. Lande, R. W. Guitink and R. A. Smith, Phys. Rev. **B 31**, 403 (1985).

[9] G. Senger, M. L. Ristig, C. E. Campbell and J. W. Clark Ann. Phys. (N.Y.) **218**, 160 (1992).

[10] M. L. Ristig, G. Senger, M. Serhan and J. W. Clark Ann. Phys. (N.Y.) **243**, 247 (1995).

[11] D. M Ceperley and E. L. Pollack, Phys. Rev. Lett. **56**, 351 (1986) and Phys. Rev. **B36**, 8343 (1987).

[12] D. M. Ceperley, Rev. Mod. Phys. **67**, 279 (1995).

[13] M. L. Ristig, T. Lindenau, M. Serhan and J. W. Clark , J. Low Temp. Phys. **114**, 317 (1999).

[14] V. Dohm, J. Low Temp. Phys. **69**, 119 (1987).

[15] J. S. Langer, Phys. Rev. **167**, 183 (1968) and **184**, 219 (1969)

[16] J. R. Johnston, Amer. J. Phys. **38**, 516 (1970).

[17] Y. Kano, J. Phys. Soc. Japan **36**, 649 (1974) and **37**, 310 (1974).

[18] J. C. Lee, Physica **116A**, 604 (1982).

[19] W. L. McMillan, Phys. Rev. A **138**, 442 (1965).

[20] S. M. Barnett and D. T. Pegg, Journal of Modern Optics **36**, 7 (1989).

[21] T. Fliessbach, Phys. Rev. **B 59**, 4334 (1999).

[22] R. Blendowske and T. Fliessbach, J. Phys.: Condens. Matter **4**, 3361 (1992).

[23] G. Ahlers, Phys. Rev. **A 8**, 530 (1973).

[24] D. S. Greywall and G. Ahlers, Phys. Rev. A **7**, 2145 (1973).

[25] V. I. Panov and V. A. Khvostikov, Sov. Phys. JETP **56**, 99 (1982).

[26] J. A. Lipa and T. C. P. Chiu, Phys. Rev. Lett. **51**, 2291 (1983).

[27] J. A. Lipa, D. R. Swanson, J. A. Nissen, T. C. P. Chiu and U. E. Israelsson, Phys. Rev. Lett. **76**, 944 (1996).

[28] K. C. Lee, R. P. Puff, Phys. Rev. **158**, 170 (1967).

[29] S. Kronauer, *Untersuchung der thermodynamischen Eigenschaften von Helium-4 am λ-Übergang*, diploma thesis, Siegen 1989.

[30] M. J. Buckingham and W. M. Fairbanks, Prog. Low Temp. Phys. **3**, p. 80, ed. by Gorter (North Holland 1961).

[31] V. D. Arp, J. Low Temp. Phys. **79**, 93 (1990).

[32] A. Singsaas, G. Ahlers, Phys. Rev. **B 29**, 4951 (1984).

[33] R. Schäfer and T. Fliessbach, Phys. Lett. **A 187**, 88 (1994).

[34] O. Penrose and L. Onsager, Phys. Rev. **104**, 576 (1956).

[35] P. A. Withlock and R. M. Panoff, Can. J. Phys. **65**, 1409 (1987).

[36] P. Nozière, *The Theory of Quantum Liquids, Volume II: Superfluid Bose Liquids* (Addison-Wesley, Redwood City, 1990).

[37] W.-M. Zhang, da H. Feng and R. Gilmore, Rev. Mod. Phys. **62**, 867 (1990).

[38] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic Press, New York 1980).