The two-fluid model with superfluid entropy

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Abstract

The two-fluid model of liquid helium is generalized to the case that the superfluid fraction has a small entropy content. We present theoretical arguments in favour of such a small superfluid entropy. In the generalized two-fluid model various sound modes of He II are investigated. In a superleak carrying a persistent current the superfluid entropy leads to a new sound mode which we call sixth sound. The relation between the sixth sound and the superfluid entropy is discussed in detail.

PACS numbers: 67.40.Bz, 67.40.Pm, 67.40.Kh

Published in Nuovo Cimento **16 D** (1994) 373

1 Introduction

The two-fluid model can be considered as the fundamental theory for the hydrodynamics of liquid helium below the λ-point. One of the model assumptions is that the entropy of the superfluid fraction vanishes. A small superfluid entropy $S_{\rm s}$ (of the order of one percent of the total entropy S) is, however, not excluded by the experiment. In this paper we generalize the two-fluid model to the case of a small superfluid entropy and investigate the consequences.

The paper *Entropy of the Superfluid Component of Helium* by Glick and Werntz [1] finds that the ratio $S_{\rm s}/S$ is less than 3%; earlier experiments cited in Ref. [1] are less accurate. A newer experiment which is sensitive to this point has been performed by Singsaas and Ahlers [2]. In this work, second sound measurements are interpreted as that of entropy. As a result, no difference to the true (caloric) entropy has been found. Due to the uncertainty in the absolute values of the caloric entropy (about 1 to 2%) the limit for $S_{\rm s}/S$ is not much lower than that of Ref. [1].

In nearly all theoretical approaches $S_{\rm s}$ is taken to be zero. As discussed by Putterman [3] this is an *assumption*. Microscopically, $S_{\rm s} = 0$ follows from the identification of the superfluid density $\rho_{\rm s}$ with the square of a macroscopic wave function [4]. This identification is plausible but unproven. In section 1.2 we present theoretical arguments in favour of a small superfluid entropy.

The remainder of this paper is based on the hypothesis of a non-vanishing superfluid entropy of a size which is not excluded by the experiment. We generalize the two-fluid model for this case (section 2). In the generalized two-fluid model we determine various sound modes (section 3). We find a new sound mode (sixth sound) which exists only if the superfluid entropy does not vanish. Section 4 discusses in detail an experiment by which the sixth sound could be detected. If the 6th sound exists this experiment would determine the superfluid entropy $S_{\rm s}$. If the 6th sound does not exist this experiment would yield a considerably lower upper limit for $S_{\rm s}$.

1.1 Two-fluid model

We start with a short review of the two-fluid model. Based on ideas by Tisza the two-fluid model was developed by Landau [5]. We use Putterman's monograph [3] as a standard reference for this model. Without dissipative terms the two-fluid equations read

$$\partial_t \rho + \boldsymbol{\nabla}(\rho_{\rm n}\,\mathbf{u}_{\rm n} + \rho_{\rm s}\,\mathbf{u}_{\rm s}) = 0\,, \tag{1}$$

$$\partial_t(\rho s) + \boldsymbol{\nabla}(\rho s\,\mathbf{u}_{\rm n}) = 0\,, \tag{2}$$

$$\partial_t(\rho_{\rm n}\,\mathbf{u}_{\rm n} + \rho_{\rm s}\,\mathbf{u}_{\rm s})_i + \partial_j(P\,\delta_{ij} + \rho_{\rm n}\,u_{{\rm n}i}u_{{\rm n}j} + \rho_{\rm s}\,u_{{\rm s}i}u_{{\rm s}j}) = 0\,, \tag{3}$$

$$m\,\partial_t\mathbf{u}_{\rm s} + m(\mathbf{u}_{\rm s}\boldsymbol{\nabla})\,\mathbf{u}_{\rm s} = -\boldsymbol{\nabla}\mu\,. \tag{4}$$

We use the abbreviations $\partial_t = \partial/\partial t$ and $\partial_i = \partial/\partial x_i$. In eq. (3) we sum over the index j. The entropy per particle is denoted by $s = S/N$, the mass of a helium atom by m, the mass density by $\rho = mN/V$, the velocity field by $\mathbf{u}$, the pressure by P and the chemical potential by μ. The indices n and s refer to the normal and superfluid phase, respectively.

Eq. (1) represents the mass conservation, eq. (2) the entropy conservation (for reversible processes), eq. (3) the momentum conservation or Euler equation, and eq. (4) the motion of the superfluid. The eight equations (1) – (4) describe the dynamics of eight independent macroscopic variables. As independent variables one might chose the temperature T, the pressure P and the velocities $\mathbf{u}_{\rm n}$ and $\mathbf{u}_{\rm s}$. These variables are fields depending on the coordinate $\mathbf{r}$ and on the time t.

All macroscopic quantities X are functions of the eight variables. Galilean invariant macroscopic quantities can be written as a function of three variables only,

$$X = X(T, P, w^2) \quad \text{where} \quad \mathbf{w} = \mathbf{u}_{\rm n} - \mathbf{u}_{\rm s}\,. \tag{5}$$

Most results (like the sound velocities) are eventually expressed by equilibrium quantities $X(T, P, 0)$ depending on T and P only.

The eqs. (1) – (4) are supplemented by dissipative terms and by Onsager's quantization rule. The resulting two-fluid model is the fundamental theory for the hydrodynamics of He II. We restrict ourselves mainly to eqs. (1) – (4) for which we discuss the modification due to $S_{\rm s} \neq 0$.

The two-fluid model is a phenomenological macroscopic theory [3]. It is based on macroscopic conservation laws (like mass conservation and the first and second law of thermodynamics) and on experimental evidence (like $S_{\rm s} \approx 0$). There are, however, also theoretical ideas on a microscopic level which support these equations. We refer in particular to London's postulate [4, 6] that the superfluid consists of a macroscopic number of particles moving coherently in a single quantum state. This conception explains that the superfluid entropy vanishes which is implicitly assumed in (2). Furthermore, it implies that the superfluid velocity is a gradient field, or that

$$\operatorname{curl}\mathbf{u}_{\rm s} = 0\,. \tag{6}$$

This statement is contained in (4). The condition (6) may be violated in a vortex. For a vortex line London's macroscopic wave function leads to Onsager's quantization rule.

1.2 Superfluid entropy

In this subsection we present arguments in favour of a small superfluid entropy $S_{\rm s}$. These arguments lead to a theoretical estimate of $S_{\rm s}$.

The non-existence of the λ-transition in ^{3}He proves that this transition is due to the exchange symmetry of the ^{4}He-atoms. The interatomic forces are about the same in liquid ^{3}He and ^{4}He; they are not the cause of the λ-transition. These facts suggest a close connection between the Bose-Einstein-condensation of the ideal Bose gas (IBG) and the λ-transition of liquid ^{4}He. This point of view has been put forward by London [7] and has subsequently been supported by several authors [8, 3].

The condensate of the IBG forms a macroscopic wave function and provides thus a model for the superfluid motion. At this point there is, however, a serious discrepancy between liquid helium and the IBG. The critical behaviour of the condensate fraction of the IBG is

$$\frac{\rho_0}{\rho} \propto |t|^{2\beta}, \qquad \beta = \frac{1}{2}\,, \tag{7}$$

where $t = (T - T_\lambda)/T_\lambda$ is the relative temperature. In contrast to (7) the superfluid fraction of He II behaves roughly like

$$\frac{\rho_{\rm s}}{\rho} \propto |t|^{2\nu}, \qquad \nu \approx \frac{1}{3}\,. \tag{8}$$

Just below the transition this implies $\rho_0 \ll \rho_{\rm s}$. The critical exponent $\beta = 1/2$ is an essential feature of the IBG. It does not appear possible to change this value by some modification of the IBG. (Note that the IBG free energy is not a logical starting point for a renormalization procedure because it already applies to an infinite system.)

We present now a possible scheme which reconciles (7) with (8) preserving at the same time the essential features of the IBG (like $\beta = 1/2$).

Following Chester [9] we use the IBG wave function together with a Jastrow factor $F = \Pi f_{ij}$; such an ansatz is based on Feynman's discussion [8]. Allowing for a condensate motion the many-body wave function reads

$$\Psi = \mathcal{S} F\,[\,\exp({\rm i}\Phi)]^{n_0} \prod_{\mathbf{k}\neq 0} [\,\varphi_{\mathbf{k}}\,]^{n_k}\,. \tag{9}$$

Here $\mathcal{S}$ denotes the symmetrization operator. The $\varphi_{\mathbf{k}}$ are the real single particle functions of the non-condensed particles and the n_k are the occupation numbers. The schematic notation $[\varphi_{\mathbf{k}}]^{n_k}$ stands for the product $\varphi_{\mathbf{k}}(\mathbf{r}_1)\cdot\varphi_{\mathbf{k}}(\mathbf{r}_2)\cdot\ldots\cdot\varphi_{\mathbf{k}}(\mathbf{r}_{n_k})$; this notation applies also to $[\exp({\rm i}\Phi)]^{n_0}$. All n_0 particles adopt the same phase factor $\exp({\rm i}\Phi(\mathbf{r}))$ forming the macroscopic wave funtion

$$\psi(\mathbf{r}) = \sqrt{\frac{n_0}{V}}\;\exp({\rm i}\Phi(\mathbf{r}))\,. \tag{10}$$

This implies that the condensate particles move coherently with the (small) velocity $\mathbf{u}_{\rm s} = \hbar\,\boldsymbol{\nabla}\Phi/m$.

Eqs. (9) and (10) are a standard description [4] for a superfluid motion in an IBG-like model. (Actually, one has to construct a suitable coherent state [10] instead of (9). This point is, however, not essential for the following discussion.) In this description the superfluid fraction $\rho_{\rm s}/\rho$ equals the condensate fraction $n_0/N = \rho_0/\rho$. We note that the current $\rho_0\,\mathbf{u}_{\rm s}$ is not depleted by the real Jastrow factors (in contrast to the condensate density ρ_0 itself).

In order to dissolve the discrepancy between (7) and (8) we assume that *non-condensed particles move coherently with the condensate*. This is possible if non-condensed particles adopt the macroscopic phase of the condensate:

$$\Psi = \mathcal{S} F \,[\,\exp(\mathrm{i}\Phi)]^{n_0} \prod_{k\le k_{\rm c}} [\,\varphi_{\mathbf{k}}\,\exp(\mathrm{i}\Phi)\,]^{n_k} \prod_{k> k_{\rm c}} [\,\varphi_{\mathbf{k}}\,]^{n_k}\,. \tag{11}$$

For the single particle states with $n_k \gg 1$ this phase ordering requires only a very small entropy change. The ansatz (11) assumes therefore this phase ordering for low momentum states. The limit $k_{\rm c}$ up to which the particles move coherently may be considered as a model parameter.

We evaluate the quantum mechanical expectation value $\langle\Psi|\,\widehat{j}\,|\Psi\rangle$ of the current operator $\widehat{j}$ for (11). The result is proportional to $\mathbf{u}_{\rm s} = \hbar\,\boldsymbol{\nabla}\Phi/m$; neither the real Jastrow factor nor the real single particle functions contribute. Equating the statistical expectation value of this current with $\rho_{\rm s}\,\mathbf{u}_{\rm s}$ yields

$$\frac{\rho_{\rm s}}{\rho} = \frac{\rho_0}{\rho} + \frac{1}{N}\sum_{k\le k_{\rm c}} \langle n_k\rangle\,. \tag{12}$$

For $\langle n_k\rangle$ we use the occupation numbers of the IBG-form. Fitting (12) to the experimental superfluid density determines $k_{\rm c}(t)$. In this way the discrepancy between (7) and (8) is removed. The asymptotic behaviour $\rho_{\rm s} \propto |t|^{2/3}$ implies $k_{\rm c} \propto |t|^{2/3}$.

With (12) fitted to the experimental superfluid density, the entropy of the contibuting non-condensed particles can be calculated. The resulting prediction [11] of this superfluid entropy $S_{\rm s}$ is shown in Fig. 1. Due to $k_{\rm c} \propto |t|^{2/3}$ the superfluid entropy per particle, $s_{\rm s} = S_{\rm s}/N_{\rm s} \propto k_{\rm c}^2 \propto |t|^{4/3}$, vanishes for $T \to T_\lambda$,

$$s_{\rm s}(T_\lambda, P) = 0\,. \tag{13}$$

Because of $\rho_0/\rho \to 1$ it vanishes also for $T \to 0$.

The experimental superfluid fraction $\rho_{\rm s}/\rho$ obeys rather well the law of corresponding states [12]; this means that it can be written as a function of $t = t(T,P) = T/T_\lambda(P) - 1$ alone. Since $s_{\rm s}$ is determined by a fit to $\rho_{\rm s}/\rho$ this should also hold for the superfluid entropy,

$$s_{\rm s}(T,P) \approx g(t) = g(T/T_\lambda(P) - 1)\,. \tag{14}$$

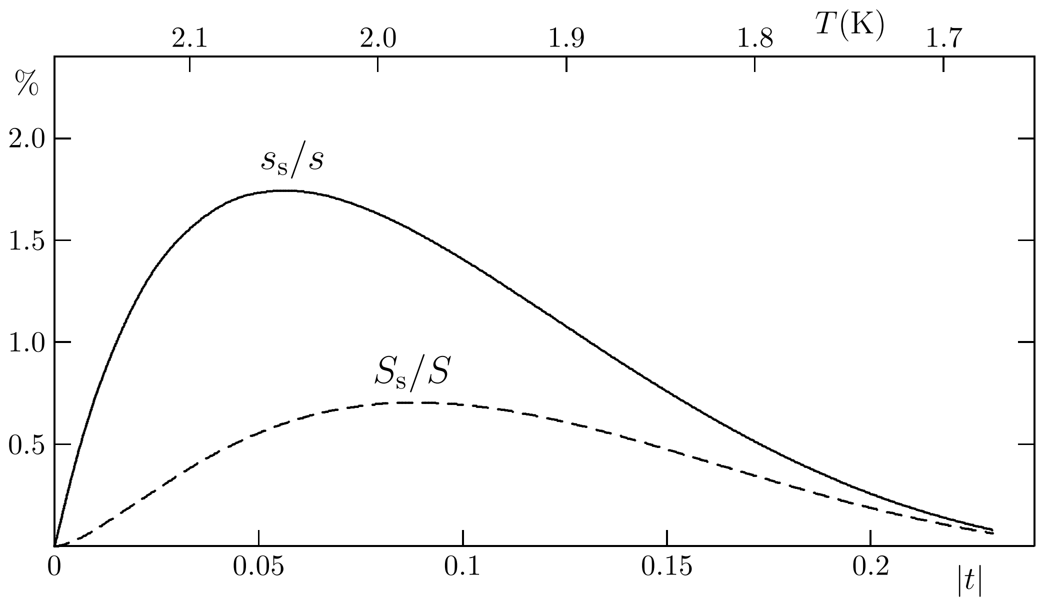

Figure 1: Prediction [11] of the superfluid density $S_s(T, P_{SVP})$ as a function of the temperature and at saturated vapour pressure.

In this way the full T- and P-dependence follows from the prediction shown in Fig. 1 (using the lower scale only). The superfluid entropy (14) is of the form (5) with $w = 0$.

The presented argument for a small superfluid entropy is based on the obvious relation between the Bose-Einstein-condensation and the λ-transition, together with the discrepancy between (7) and (8). This argument serves as a motivation of our investigation. The remainder of this paper could just as well be based on the mere hypothesis of a non-vanishing superfluid entropy.

2 Modified two-fluid model

In this section we generalize the two-fluid model to the case of a small superfluid entropy $s_s = s_s(T, P, w^2)$.

2.1 Mass, momentum and entropy conservation

In eqs. (1) and (3) the normal and the superfluid part are treated in a symmetric way. Therefore these equations are unchanged by a non-vanishing superfluid entropy:

$$\partial_t \rho + \boldsymbol{\nabla}(\rho_n \mathbf{u}_n + \rho_s \mathbf{u}_s) = 0 \quad \text{for } s_s \neq 0\,, \tag{15}$$

$$\partial_t (\rho_n \mathbf{u}_n + \rho_s \mathbf{u}_s)_i + \partial_j (P\,\delta_{ij} + \rho_n u_{ni} u_{nj} + \rho_s u_{si} u_{sj}) = 0 \quad \text{for } s_s \neq 0\,. \tag{16}$$

In eq. (2) the entropy density ρs is carried by the normal fraction alone; consequently, the entropy current density is $\rho s\, \mathbf{u}_n$. For $s_s \neq 0$ the entropy density is partly carried by the superfluid fraction leading to a different entropy current density:

$$\rho s\,\mathbf{u}_{\rm n} \;\longrightarrow\; \rho_{\rm n}\,s_{\rm n}\,\mathbf{u}_{\rm n} + \rho_{\rm s}\,s_{\rm s}\,\mathbf{u}_{\rm s}\,. \tag{17}$$

Here $s_{\rm s} = S_{\rm s}/N_{\rm s}$ and $s_{\rm n} = (S - S_{\rm s})/(N - N_{\rm s})$. The entropy continuity equation reads now

$$\partial_t\,(\rho s) + \boldsymbol{\nabla}\,(\rho_{\rm n}\,s_{\rm n}\,\mathbf{u}_{\rm n} + \rho_{\rm s}\,s_{\rm s}\,\mathbf{u}_{\rm s}) = 0 \quad \text{for } s_{\rm s} \neq 0\,. \tag{18}$$

2.2 Equation of superfluid motion

In the static limit eq. (4) becomes $\boldsymbol{\nabla}\mu = -s\,\boldsymbol{\nabla}T + v\,\boldsymbol{\nabla}P = 0$ where $v = V/N = m/\rho$. This yields the well-known fountain pressure (FP)

$$\left(\frac{dP}{dT}\right)_{\rm FP} = \frac{s}{v} \quad \text{for } s_{\rm s} = 0\,. \tag{19}$$

The origin of the FP may be explained as follows: Consider two containers with liquid He II connected by a superleak; initially both containers have the same temperature T and pressure P. By increasing the pressure in one container some superfluid liquid is pushed through the superleak to the other container. Since the superfluid fraction carries no entropy the other container becomes colder; effectively $dP > 0$ in the first container is accompanied by $dT > 0$. According to (19) the ratio dP/dT is proportional to the *missing* entropy s per transferred particle. A possible superfluid entropy $s_{\rm s} = S_{\rm s}/N_{\rm s}$ diminishes the missing entropy per transferred particle, which means

$$s \;\longrightarrow\; s - s_{\rm s} \qquad \text{(entropy deficit of a superfluid particle relative to average).} \tag{20}$$

As in (17), this replacement reflects the change in the entropy transport due to $s_{\rm s} \neq 0$. The replacement (20) applies, however, to the static limit and does not immediately yield the wanted generalization of (4).

A basic property of the superfluid motion is $\text{curl}\,\mathbf{u}_{\rm s} = 0$, eq. (6). This property is well-established experimentally. Theoretically, it follows from the conception that the supervelocity is the gradient of a macroscopic phase; this conception is not altered by the modified picture presented in section 1.2. Eq. (6) implies that the l.h.s. of (4) is a gradient field. Therefore, the generalization of (4) must be of the form

$$m\,\partial_t\,\mathbf{u}_{\rm s} + m\,(\mathbf{u}_{\rm s}\boldsymbol{\nabla})\,\mathbf{u}_{\rm s} = -\boldsymbol{\nabla}(\mu - \mu_{\rm s}) \quad \text{for } s_{\rm s} \neq 0\,. \tag{21}$$

The chemical potential μ yields the contribution $-\partial\mu/\partial T = s$ in (19). Consequently, the replacement (20) implies

$$\frac{\partial\mu_{\rm s}(T,P,w^2)}{\partial T} = -s_{\rm s}(T,P,w^2)\,. \tag{22}$$

Together with $s_{\rm s}$, eq. (13), the discussed modifications should vanish at T_λ. Therefore

$$\mu_{\rm s}(T,P,w^2) = -\int_{T_\lambda}^{T} dT'\; s_{\rm s}(T',P,w^2)\,. \tag{23}$$

Eq. (21) with (23) defines the generalization of (4).

We determine the FP from the generalized equation of motion (21). The FP experiment is done for $\mathbf{u}_{\rm n} = \mathbf{u}_{\rm s} = 0$ or $w^2 = 0$. Then eq. (21) yields $\boldsymbol{\nabla}(\mu - \mu_{\rm s}) = 0$ and

$$\left(\frac{dP}{dT}\right)_{\rm FP} = \frac{s - s_{\rm s}}{v - v_{\rm s}} \approx \frac{s - s_{\rm s}}{v} \tag{24}$$

where $s_{\rm s} = s_{\rm s}(T,P,0)$ and

$$v_{\rm s} = \frac{\partial \mu_{\rm s}(T,P,0)}{\partial P} = -\int_{T_\lambda}^{T} dT'\; \frac{\partial s_{\rm s}(T',P,0)}{\partial P}\,. \tag{25}$$

The quantity $v_{\rm s}$ is rather small and may be neglected in (24). From (14) we obtain $T_\lambda\,(\partial s_{\rm s}/\partial P) = -T\,(\partial s_{\rm s}/\partial T)\,dT_\lambda/dP$. Using this, $dT_\lambda/dP \approx -0.01\,\mathrm{K/bar}$ and the entropy $s_{\rm s}$ of Fig. 1, the integral (25) can be evaluated numerically. The resulting $|v_{\rm s}/v|$ has a similar temperature dependence as $s_{\rm s}/s$. The absolute values of $|v_{\rm s}/v|$ are much smaller,

$$\left|\frac{v_{\rm s}}{v}\right| \le 3\cdot 10^{-4}. \tag{26}$$

The term $v_{\rm s}$ has not been considered in previous generalizations [13, 11] of (19). It is derived theoretically; it follows from $\mathrm{curl}\,\mathbf{u}_{\rm s} = 0$ and the pressure dependence of the superfluid entropy.

2.3 Summary

The two-fluid equations with $s_{\rm s} \neq 0$ are

$$\partial_t\,\rho + \boldsymbol{\nabla}(\rho_{\rm n}\,\mathbf{u}_{\rm n} + \rho_{\rm s}\,\mathbf{u}_{\rm s}) = 0\,, \tag{27}$$

$$\partial_t\,(\rho s) + \boldsymbol{\nabla}(\rho_{\rm n}\,s_{\rm n}\,\mathbf{u}_{\rm n} + \rho_{\rm s}\,s_{\rm s}\,\mathbf{u}_{\rm s}) = 0\,, \tag{28}$$

$$\partial_t\,(\rho_{\rm n}\,\mathbf{u}_{\rm n} + \rho_{\rm s}\,\mathbf{u}_{\rm s})_i + \partial_j\,(P\,\delta_{ij} + \rho_{\rm n}\,u_{{\rm n}i}\,u_{{\rm n}j} + \rho_{\rm s}\,u_{{\rm s}i}\,u_{{\rm s}j}) = 0\,, \tag{29}$$

$$m\,\partial_t\,\mathbf{u}_{\rm s} + m\,(\mathbf{u}_{\rm s}\boldsymbol{\nabla})\,\mathbf{u}_{\rm s} = -\boldsymbol{\nabla}(\mu - \mu_{\rm s}) \tag{30}$$

where

$$\mu_{\rm s}(T,P,w^2) = -\int_{T_\lambda}^{T} dT'\; s_{\rm s}(T',P,w^2)\,. \tag{31}$$

For $s_s = 0$ these equations reduce to (1) – (4). They are still eight equations for eight variables. The superfluid entropy $s_{\rm s}$ is just a further macroscopic quantity; as any other Galilean invariant macroscopic quantity it is of the form (5). It does not constitute a new independent variable.

In addition we note:

1. The equations (27) – (30) have to be supplemented by terms describing dissipative effects. These terms might also contain corrections of the size $\mathcal{O}(s_\mathrm{s}/s)$ relative to their well-known form [3]. For practical purposes (like an estimate of the damping of sound modes) we will use unmodified dissipative terms.

2. Eq. (6) follows from (30). Onsager's quantization rule is unchanged.

Together with the two-fluid equations the underlying microscopic conception is slightly modified (section 1.2). As usual there is a macroscopic wave function $\psi = \sqrt{\rho_0}\,\exp(\mathrm{i}\,\Phi(\mathbf{r}))$ which determines the supervelocity $\mathbf{u}_\mathrm{s} = \hbar\,\boldsymbol{\nabla}\Phi/m$. What is given up is the identification of $|\psi|^2$ with ρ_s. At the same time London's main point, namely the relation between the IBG and liquid helium, is reinforced by reconciling (7) with (8).

3 Sound modes

3.1 Introduction

A major and important application of hydrodynamic equations is the evaluation of sound modes. This application is of particular interest for testing the modified two-fluid model because sound velocities can be measured with high accuracy. We consider the first and second sound of bulk He II (section 3.2) and the fourth sound of clamped He II (section 3.3). We determine the corrections in the sound velocities due to a non-vanishing superfluid entropy. The derivation of the fourth sound for $s_\mathrm{s} \neq 0$ leads nearly automatically to a new sound mode which we call sixth sound. The detailed calculations are given in Ref. [14]. The sixth sound has already been presented in a short letter [15].

The standard ansatz for sound modes

$$T(\mathbf{r},t) = T_0 + \Delta T\,\exp(\mathrm{i}(\mathbf{kr}-\omega t))\,, \tag{32}$$

$$P(\mathbf{r},t) = P_0 + \Delta P\,\exp(\mathrm{i}(\mathbf{kr}-\omega t))\,, \tag{33}$$

$$\mathbf{u}_\mathrm{n}(\mathbf{r},t) = \mathbf{u}_{\mathrm{n},0} + \Delta\mathbf{u}_\mathrm{n}\,\exp(\mathrm{i}(\mathbf{kr}-\omega t))\,, \tag{34}$$

$$\mathbf{u}_\mathrm{s}(\mathbf{r},t) = \mathbf{u}_{\mathrm{s},0} + \Delta\mathbf{u}_\mathrm{s}\,\exp(\mathrm{i}(\mathbf{kr}-\omega t)) \tag{35}$$

is inserted into (27) – (30). The constant values T_0, P_0, $\mathbf{u}_{\mathrm{n},0}$ and $\mathbf{u}_{\mathrm{s},0}$ solve the equations. Quadratic and higher order terms in the (small) amplitudes ΔT, ΔP, $\Delta\mathbf{u}_\mathrm{n}$ and $\Delta\mathbf{u}_\mathrm{s}$ are omitted. This yields a linear, homogeneous system of equations for the amplitudes. For a non-trivial solution the determinant of the coefficient matrix must vanish. This condition yields solutions of the form $\omega_\nu = \omega_\nu(k)$ (for a specific direction of $\mathbf{k}$). The ratio

$$c_\nu = \frac{\omega_\nu(k)}{k} \tag{36}$$

is the velocity of the sound wave. The sound velocities of the common two-fluid model (1) – (4) are denoted by $c_{\nu,0}$. Because of

$$\frac{s_\mathrm{s}}{s} \le 2 \cdot 10^{-2}\,, \qquad \left|\frac{v_\mathrm{s}}{v}\right| \le 3 \cdot 10^{-4} \tag{37}$$

the differences between c_ν and $c_{\nu,0}$ are expected to be small.

All coefficients in the linearized equations are taken at T_0, P_0 and $w_0^2 = (\mathbf{u}_{\mathrm{n},0} - \mathbf{u}_{\mathrm{s},0})^2$. The final results are expressed by thermodynamic quantities at $w_0 = 0$. Eventually we omit the index zero and use the notation

$$X(T, P) = X(T_0, P_0, 0)\,. \tag{38}$$

3.2 First and second sound

The equilibrium state of bulk He II has a given temperature T_0, pressure P_0 and

$$\mathbf{u}_{\mathrm{n},0} = \mathbf{u}_{\mathrm{s},0} = 0\,. \tag{39}$$

We insert (32) – (35) into (27) – (30) and determine (36). This calculation is quite analogous to that [3, 6] in the common two-fluid model. Therefore, we restrict ourselves to a presentation of the results. Because of (39) the coefficients in the linearized equations and the sound velocities are of the form (38).

For the first and second sound the leading corrections to $c_{1,0}$ and $c_{2,0}$ are given by

$$c_1(T, P) = c_{1,0}\left(1 - \frac{s_\mathrm{s}}{s}\left(1 - \frac{c_V}{c_P}\right)\frac{u_2^2}{u_1^2} - \frac{v_\mathrm{s}}{v}\,\frac{T\,(s_\mathrm{n} - s)}{2\rho\, c_P}\left(\frac{\partial\rho}{\partial T}\right)_P\right) \tag{40}$$

and

$$c_2(T, P) = c_{2,0}\left(1 - \frac{s_\mathrm{s}}{s}\right)\,. \tag{41}$$

We use the abbreviations

$$u_1 = \sqrt{\frac{\partial P(\rho, s)}{\partial \rho}} \quad \text{and} \quad u_2 = \sqrt{\frac{T\,\rho_\mathrm{s}\, s^2}{m\, \rho_\mathrm{n}\, c_V}}\,. \tag{42}$$

By c_V and c_P we denote the specific heats at constant volume and pressure, respectively. The standard approximations for $c_{1,0}$ and $c_{2,0}$ are

$$c_{1,0} \approx u_1 \quad \text{and} \quad c_{2,0} \approx u_2 \sqrt{\frac{c_V}{c_P}}\ . \tag{43}$$

The corrections to these approximations are discussed in detail in Ref. [3]; they are of the relative order $(u_2^2/u_1^2)(1 - c_V/c_P) \sim 10^{-5}$.

There are two correction terms in (40). The first one has the relative size $2 \cdot 10^{-7}$, the second one $2 \cdot 10^{-8}$. These corrections are of academic interest only; they are even smaller than the error in $c_{1,0} \approx u_1$.

The corrections for the second sound are of the order $s_{\rm s}/s \le 2\%$; they might be observable. Using (24) we may define a 'fountain pressure' entropy $s_{\rm FP}$ by

$$\frac{s_{\rm FP}}{v} = \left(\frac{dP}{dT}\right)_{\rm FP}\ . \tag{44}$$

This yields a common expression for c_2 and $c_{2,0}$:

$$\sqrt{\frac{T\,\rho_{\rm s}\,s_{\rm FP}^2}{m\,\rho_{\rm n}\,c_P}} = \begin{cases} c_{2,0} & (s_{\rm s} = 0) \\ c_2 & (s_{\rm s} \ne 0)\,. \end{cases} \tag{45}$$

3.3 Fourth and sixth sound

In this subsection we consider He II in which the normal phase is clamped,

$$\mathbf{u}_{\rm n}(\mathbf{r}, t) = 0\,. \tag{46}$$

This reduces the number of variables to five, for which we may choose T, P and $\mathbf{u}_{\rm s}$. Their dynamics is determined by the five equations (27), (28) and (30); eq. (29) is effectively replaced by (46).

A standard device for measuring sound modes in clamped helium is a ring filled with powder (Fig. 2). The position (middle line) of the ring may be described by

$$\mathbf{r}_{\rm ring} = (R\cos\phi,\ R\sin\phi,\ 0)\,, \qquad \phi = 0, \ldots, 2\pi\,. \tag{47}$$

The thickness of the ring is assumed to be small compared to the radius R. Then the $\mathbf{r}$-dependence of the considered fields reduces to a ϕ-dependence and the supervelocity is parallel to the ring,

$$\mathbf{u}_{\rm s}(\mathbf{r}, t) = u_{\rm s}(\phi, t)\, \mathbf{e}_\phi. \tag{48}$$

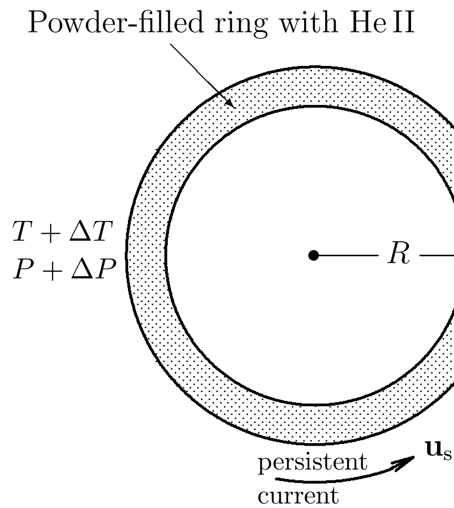

Figure 2: A ring with clamped helium is a standard device for fourth sound and persistent current experiments. We consider the solutions of the linearized equations of motion for this configuration. For $s_s = 0$ these solutions are the fourth sound and a static fountain pressure (FP) gradient $\Delta P/\Delta T$. For $s_s \neq 0$ (and in the presence of a persistent current) the FP gradient becomes a sound mode which we call sixth sound.

This reduces the number of variables and equations from five to three. Using (46) and (48) the field equations (27), (28) and (30) become

$$\partial_t \rho + \frac{1}{R}\frac{\partial}{\partial\phi}(\rho_s u_s) = 0, \tag{49}$$

$$\partial_t(\rho s) + \frac{1}{R}\frac{\partial}{\partial\phi}(\rho_s s_s u_s) = 0, \tag{50}$$

$$m\,\partial_t u_s + \frac{1}{R}\frac{\partial}{\partial\phi}\left(\mu - \mu_s + \frac{m u_s^2}{2}\right) = 0. \tag{51}$$

The following derivation is simplified by using the variables

$$T, \quad \mu_c = \mu - \mu_s + \frac{m u_s^2}{2} \quad \text{and} \quad j = \rho_s u_s \tag{52}$$

rather than T, P and u_s. Taking into account (46), the specific geometry, and the new variables, the ansatz for sound modes reads

$$T(\phi,t) = T_0 + \Delta T \exp(\mathrm{i}[kR\phi - \omega t]), \tag{53}$$

$$\mu_c(\phi,t) = \mu_{c,0} + \Delta\mu_c \exp(\mathrm{i}[kR\phi - \omega t]), \tag{54}$$

$$j(\phi,t) = j_0 + \Delta j \exp(\mathrm{i}[kR\phi - \omega t]). \tag{55}$$

The equilibrium state has the constant values T_0, $\mu_{c,0}$ and

$$j_0 = \rho_{s,0} u_{s,0} = \text{const.}. \tag{56}$$

A persistent current ($j_0 \neq 0$) is a metastable equilibrium.

Because of $T(\phi, t) = T(\phi + 2\pi, t)$ the possible k-values are restricted to

$$k = \frac{n}{R} \quad \text{where} \quad n \in \{\pm 1, \pm 2, \pm 3, \ldots\}. \tag{57}$$

Inserting (53) – (55) into (49) – (51), the resulting linearized equations are:

$$\begin{pmatrix} \omega\,\dfrac{\partial \rho}{\partial T} & \omega\,\dfrac{\partial \rho}{\partial \mu_{\mathrm{c}}} & \omega\,\dfrac{\partial \rho}{\partial j} - k \\ \omega\,\rho\,\dfrac{\partial s}{\partial T} - k\,j\,\dfrac{\partial s_{\mathrm{s}}}{\partial T} & \omega\,\rho\,\dfrac{\partial s}{\partial \mu_{\mathrm{c}}} - k\,j\,\dfrac{\partial s_{\mathrm{s}}}{\partial \mu_{\mathrm{c}}} & \omega\,\rho\,\dfrac{\partial s}{\partial j} + (s - s_{\mathrm{s}})k - k\,j\,\dfrac{\partial s_{\mathrm{s}}}{\partial j} \\ \omega\,u_{\mathrm{s}}\,\dfrac{\partial \rho_{\mathrm{s}}}{\partial T} & \omega\,u_{\mathrm{s}}\,\dfrac{\partial \rho_{\mathrm{s}}}{\partial \mu_{\mathrm{s}}} + \rho_{\mathrm{s}}\,k & \omega\,u_{\mathrm{s}}\,\dfrac{\partial \rho_{\mathrm{s}}}{\partial j} - \omega \end{pmatrix} \begin{pmatrix} \Delta T \\ \Delta \mu_{\mathrm{c}} \\ \Delta j \end{pmatrix} = 0\,. \tag{58}$$

The first line represents eq. (49), the third line eq. (51). For the second line we subtracted eq. (49) times s from (50). All coefficients in (58) are to be taken at T_0, $\mu_{\mathrm{c},0}$ and j_0. Here and in the following we suppress the index zero. In each partial derivative two of the three variables T, μ_{c} and j are kept constant.

For a non-trivial solution the determinant of the coefficient matrix in (58) must vanish. This condition yields three eigenvalues ω_ν (for given k), or three sound velocities c_ν. The three sound velocities may be distinguished by their dependence on the velocity u_{s}:

$$c_\nu = \begin{cases} \pm c_4 & \propto \quad 1 + \mathcal{O}(u_{\mathrm{s}}) \\ c_6 & \propto \quad u_{\mathrm{s}}\,(1 + \mathcal{O}({u_{\mathrm{s}}}^2))\,. \end{cases} \tag{59}$$

The solutions $\pm c_4$ are those of the well-known fourth sound. For $s_{\mathrm{s}} \neq 0$ and $u_{\mathrm{s}} \neq 0$ we obtain a new sound mode which we call *sixth sound*.

In the following we restrict ourselves to the leading order in u_{s}. Higher order contributions are necessarily small because $|u_{\mathrm{s}}|$ must be less than the critical velocity u_{crit}. In a specific experiment, these contributions can be further suppressed by choosing $|u_{\mathrm{s}}| \ll u_{\mathrm{crit}}$. The coefficients appearing in the linearized equations are written as

$$X(T, \mu_{\mathrm{c}}, j^2) = X(T, \mu_{\mathrm{c}}) + \mathcal{O}(u_{\mathrm{s}}^2)\,. \tag{60}$$

The condition for a non-trivial solution of (58) yields

$$c_4 = \sqrt{\frac{\rho_{\mathrm{s}}}{m\rho}\;\frac{(s - s_{\mathrm{s}})\,\dfrac{\partial \rho}{\partial T} + \rho\,\dfrac{\partial s}{\partial T}}{\dfrac{\partial \rho}{\partial \mu_{\mathrm{c}}}\,\dfrac{\partial s}{\partial T} - \dfrac{\partial \rho}{\partial T}\,\dfrac{\partial s}{\partial \mu_{\mathrm{c}}}}} \tag{61}$$

and

$$c_6 = u_{\rm s}\,\frac{\rho_{\rm s}}{\rho}\;\frac{\dfrac{\partial s_{\rm s}}{\partial T}}{\dfrac{\partial s}{\partial T}+\dfrac{s-s_{\rm s}}{\rho}\;\dfrac{\partial \rho}{\partial T}}\;. \tag{62}$$

Since we restrict ourselves to the leading order in $u_{\rm s}$ all quantities are to be taken at T, $\mu_{\rm c}$ and $j = 0$, eq. (60). We switch now to the arguments T, P and $w^2 = 0$ by writing

$$\frac{\partial X}{\partial T} \;=\; \frac{\partial X(T,\mu_{\rm c})}{\partial T} \;=\; \left(\frac{\partial X}{\partial T}\right)_P + \frac{s-s_{\rm s}}{v-v_{\rm s}}\,\left(\frac{\partial X}{\partial P}\right)_T\,, \tag{63}$$

$$\frac{\partial X}{\partial \mu_{\rm c}} \;=\; \frac{\partial X(T,\mu_{\rm c})}{\partial \mu_{\rm c}} \;=\; \frac{1}{v-v_{\rm s}}\,\left(\frac{\partial X}{\partial P}\right)_T\;. \tag{64}$$

We use $(\partial\mu_{\rm c}/\partial T)_P = -(s - s_{\rm s})$ and $(\partial\mu_{\rm c}/\partial P)_T = v - v_{\rm s}$ which follow from (22) and (25). With (63) and (64) we obtain from (61) and (62):

$$c_4 = \sqrt{\frac{\rho_{\rm s}}{\rho}\left(1-\frac{v_{\rm s}}{v}\right)\left[1+\frac{T\,(s-s_{\rm s})}{\rho\, c_P}\,\frac{2-v_{\rm s}/v}{1-v_{\rm s}/v}\left(\frac{\partial\rho}{\partial T}\right)_P\right]u_1^2+\frac{\rho_{\rm n}}{\rho}\left(1-\frac{s_{\rm s}}{s}\right)^2 u_2^2} \tag{65}$$

and

$$c_6 = u_{\rm s}\,\frac{\rho_{\rm s}}{\rho}\;\frac{\left(\dfrac{\partial s_{\rm s}}{\partial T}\right)_P+\dfrac{s-s_{\rm s}}{v-v_{\rm s}}\left(\dfrac{\partial s_{\rm s}}{\partial P}\right)_T}{\left(\dfrac{\partial s}{\partial T}\right)_P\left[1+\dfrac{\rho_{\rm n}u_2^2\,(1-s_{\rm s}/s)^2}{\rho_{\rm s}u_1^2\,(1-v_{\rm s}/v)}\right]+\dfrac{s-s_{\rm s}}{\rho}\,\dfrac{2-v_{\rm s}/v}{1-v_{\rm s}/v}\left(\dfrac{\partial\rho}{\partial T}\right)_P}\;. \tag{66}$$

These results are exact to any order in $s_{\rm s}/s$ and $v_{\rm s}/v$, and to leading order in $u_{\rm s}$. We present now approximate but simpler formulae. Neglecting the higher order corrections eq. (65) becomes

$$c_4(T,P) = \sqrt{\frac{\rho_{\rm s}}{\rho}\left(1-\frac{v_{\rm s}}{v}\right)\left[1+\frac{2\,T\,(s-s_{\rm s})}{\rho\, c_P}\left(\frac{\partial\rho}{\partial T}\right)_P\right]u_1^2+\frac{\rho_{\rm n}}{\rho}\left(1-\frac{s_{\rm s}}{s}\right)^2 u_2^2}\;. \tag{67}$$

For $v_{\rm s} = 0$ and $s_{\rm s} = 0$ this result is well-known [3]. The first term under the square root is the dominant one; therefore it is appropriate to retain the small correction $v_{\rm s}/v$ in this term (as compared to the larger correction $s_{\rm s}/s$ in the second term). The correction in the second term corresponds to that in (41); the one in the first term is different from that in (40).

The sound velocity c_6 is proportional to $s_{\rm s}$. For a more handy expression we may therefore neglect all corrections to c_6 of the order of one percent. We go back to (62) and use $\partial s_\mathrm{s}/\partial T = (\partial s_\mathrm{s}/\partial T)_{\mu_\mathrm{c}} \approx (\partial s_\mathrm{s}/\partial T)_\mu$ and $\partial s/\partial T \approx (\partial s/\partial T)_\mu$. Because

$$\left| \frac{s}{\rho}\,\frac{\partial\rho}{\partial T} \right| \sim 10^{-2} \left| \frac{\partial s}{\partial T} \right| , \tag{68}$$

the second term in the denominator in (62) can be omitted. Thus we obtain

$$c_6(T,P) = u_\mathrm{s}\,\frac{\rho_\mathrm{s}}{\rho}\,\frac{c_{\mu,\mathrm{s}}}{c_\mu} \tag{69}$$

where

$$c_\mu \;=\; T\left(\frac{\partial s}{\partial T}\right)_\mu \quad \text{and} \quad c_{\mu,\mathrm{s}} = T\left(\frac{\partial s_\mathrm{s}}{\partial T}\right)_\mu \tag{70}$$

are the specific heats at constant chemical potential. For the practical evaluation (next section) one may use the specific heats at constant pressure instead.

3.4 Discussion of the sixth sound

The fourth sound is experimentally and theoretically well-established. The following discussion centers therefore on the sixth sound. We derive the amplitudes of this sound mode and its damping.

For $s_\mathrm{s} = 0$ the frequency of sixth sound becomes zero; the solution (32), (33) represents a static temperature and pressure gradient. For the discussion of this limit we may go back to the equations of motion (49) – (51): For $\partial_t = 0$ the first equation is solved by $\partial(\rho_\mathrm{s} u_\mathrm{s})/\partial\phi = 0$, and the second one is trivially fulfilled. The third one yields $\mu - \mu_\mathrm{s} = \text{const.} + \mathcal{O}(u_s^2)$ or $\mu \approx \text{const.}$. That means that the sixth sound reduces to a static fountain pressure (FP) gradient in the limit $s_\mathrm{s} = 0$. This situation is changed for $s_\mathrm{s} \neq 0$: The second term in (50) is now non-zero (provided that $u_\mathrm{s} \neq 0$) and leads via the first term to a time-dependence. The originally static fountain pressure gradient becomes an oscillating mode; effectively, the FP gradient is shifted with velocity c_6 along the ring. The sixth sound may be conceived as an entropy transport along the ring caused by the persistent current (section 4).

As discussed, the sixth sound amplitudes solve eq. (51) by $\mu \approx \text{const.}$. This means

$$\left(\frac{\Delta T}{\Delta P}\right)_{\text{6th sound}} \approx \frac{v}{s}\,. \tag{71}$$

The corresponding ratio for the fourth sound is well-known (for $s_\mathrm{s} = 0$). Neglecting corrections of the order 10^{-2} this ratio is

$$\left(\frac{\Delta T}{\Delta P}\right)_{\text{4th sound}} \approx \left[\frac{\rho_\mathrm{n}\,u_2^2}{\rho_\mathrm{s}\,u_1^2} + \frac{T\,s}{\rho\,c_P}\left(\frac{\partial\rho}{\partial T}\right)_P \right] \frac{v}{s} \ll \frac{v}{s}\,. \tag{72}$$

The fourth and sixth sound can be distinguished by their amplitudes and by their velocities: The temperature amplitude of the fourth sound is much smaller than that of the sixth sound (for a given pressure amplitude). The velocity of the sixth sound has a characteristic proportionality to $u_{\rm s}$.

In order to calculate the damping of the sound modes the two-fluid equations are supplemented by (unmodified) dissipative terms [3]. The detailed calculation [14] shows that only the heat conductivity κ contributes to the damping of the sixth sound. We derive this damping in a simplified way. Taking into account the heat conduction the first coefficient in the second row of (58) becomes

$$\omega\,\rho\,\frac{\partial s}{\partial T} - k\,j\,\frac{\partial s_{\rm s}}{\partial T} + {\rm i}\,\frac{m\kappa}{T}\,k^2 \approx 0\,. \tag{73}$$

For the sixth sound with its dominant ΔT-amplitude in (58) this coefficient must be approximately zero. For $\kappa = 0$ this condition yields the velocity $c_6 = \omega/k$ of (69). For $\kappa \neq 0$ we obtain instead

$$\omega = c_6\,k - {\rm i}\,\frac{mk^2\kappa}{\rho\,c_\mu} = \omega_{\rm FP} - {\rm i}\,\Gamma_{\rm FP}\,. \tag{74}$$

This result is of the same accuracy as (69). The real quantities $\omega_{\rm FP}$ and $\Gamma_{\rm FP}$ will be used and discussed in section 4.

3.5 Detectability of a superfluid entropy

Each of the calculated sound velocities has some corrections due to $s_{\rm s} \neq 0$. We discuss whether a possible superfluid entropy may be detected by measuring the sound velocity:

First sound: The relative difference between c_1 and $c_{1,0}$ is of the order 10^{-7}. This is many orders below the experimental accuracy (and also below the theoretical accuracy of $c_{1,0}$).

Second sound: The velocity c_2 has been measured with a systematic error of less than 0.4% [2]. Assuming the validity of (19) this measurement has been interpreted as that of the entropy. This entropy may be called 'fountain pressure' entropy $s_{\rm FP}$, eqs. (44) and (45). According to Fig. 1 we expect a 1 to 2% difference between $s_{\rm FP}$ and the true (caloric) entropy s. Unfortunately the absolute value of s at the λ-point is also uncertain by about 2%.

Improving the present experimental accuracy by some factor (say 5) one should be able to see a difference between s and $s_{\rm FP}$, in particular because of the steep rise of $s_{\rm s}$ just below the λ-point (Fig. 1). This possibility has been discussed in more detail in Ref. [11].

Fourth sound: The fourth sound velocity (67) is dominated by the first term under the square root. The correction in this term is of the order 10^{-4}; it is smaller than the experimental accuracy.

Sixth sound: The sixth sound velocity is proportional to the superfluid entropy. The sixth sound is therefore the prime candidate for detecting and measuring the superfluid entropy. Its observability will be discussed in detail in section 4.

The attempt of measuring $s - s_{\rm FP}$ by the second sound will not become obsolete by a possible detection of the sixth sound. The reason is that the two modes are sensitive to different modifications of the two-fluid equations. The sixth sound velocity is due to the $s_{\rm s}$-contribution in the entropy continuity equation (28); the $\mu_{\rm s}$-term in (30) yields higher order corrections to c_6 only. On the other hand, a main correction in the second sound velocity stems from the $\mu_{\rm s}$-term in (30).

4 Observability of the sixth sound

We start this section with a simplified, alternative derivation of the sixth sound. We discuss then in detail the possible observation of the sixth sound.

4.1 Entropy transport by the sixth sound

We assume that the ring of Fig. 2 carries a persistent current with the velocity $u_{\rm s}$, and that at a certain time there is the following temperature and pressure variation along the ring:

$$\delta T(\phi) = A\,\cos(n\phi)\,, \qquad \delta P(\phi) \approx \frac{s}{v}\,\delta T\,, \qquad n \in \{1, 2, \ldots\}\,. \tag{75}$$

This variation implies $\mu \approx$ const. (the small l.h.s. of (30) and the $\mu_{\rm s}$-term may be neglected for the present purpose). The variation (75) is a fountain pressure gradient; for $s_{\rm s} = 0$ it is metastable.

Eq. (75) implies a corresponding variation of the entropy density:

$$\rho\,\delta s = \frac{\rho\, c_\mu}{T}\,\delta T(\phi)\,. \tag{76}$$

Due to (68) we may use $\rho \approx$ const.. The continuity equation (49) implies then $\rho_{\rm s}\, u_{\rm s} \approx$ const..

For $s_{\rm s} \neq 0$ the persistent current carries entropy. For constant T and P the net entropy current δj_s vanishes. For (75) we obtain (using $\rho_{\rm s}\, u_{\rm s} \approx$ const.):

$$\delta j_s = \rho_{\rm s}\, u_{\rm s}\,\delta s_{\rm s} = \rho_{\rm s}\, u_{\rm s}\,\frac{c_{\mu,{\rm s}}}{T}\,\delta T(\phi)\,. \tag{77}$$

Due to this heat current the entropy variation (76) is shifted along the ring. This shift takes place with the velocity

$$c_6 = \frac{\delta j_{\rm s}}{\rho\,\delta s} = u_{\rm s}\,\frac{\rho_{\rm s}}{\rho}\,\frac{c_{\mu,{\rm s}}}{c_\mu}\,. \tag{78}$$

In this rather simple way the sixth sound may be understood as an entropy transport phenomenon. Section 3 shows that it is also – in accordance with the common nomenclature – a sound mode.

The net entropy current (77) must change the entropy variation of (76) of the *whole ring* (consisting of the container, the powder and He II). For the considered experiment, the entropy density ρs in (50) must therefore include the entropy of the powder and of the container. In practice, this can be taken into account by equating c_μ in (78) with the specific heat of the whole ring, $c_\mu = c_{\rm ring}$. At the considered low temperatures, the specific heat (per atom) of helium is at least four orders of magnitude larger than that of normal solid material. Therefore, we may set $c_{\rm ring} \approx c({\rm He})$. In addition, the specific heat at constant chemical potential may be approximated by that at constant pressure (implying an error of the order 10^{-2}):

$$c_\mu = c_{\rm ring} \approx c_P({\rm He})\,, \qquad \kappa = \kappa_{\rm ring} \approx \kappa({\rm He})\,. \tag{79}$$

These arguments apply similarily for the heat conductivity κ.

4.2 Shift of the temperature variation

Any temperature and pressure variation in an actual ring experiment is likely to have a FP component. In the presence of a persistent current this variation is shifted with the velocity c_6 along the ring. This shift should be observable. It has characteristic properties:

- The shift velocity is proportional to that of the persistent current.
- The shift velocity is independent of the amplitude of the temperature variation (as long as the linear approximation works).

The observation of the first property would leave no doubt that the persistent current transports entropy, implying that the superfluid entropy is non-zero. The experiment would then determine this superfluid entropy quantitatively. If no such shift is observed the experiment would yield an upper limit for $S_{\rm s}$ which is considerably lower than the present one.

For a potential experiment it is useful to know the expected size of the effect. We use $c_{\mu,{\rm s}} \approx c_{P,{\rm s}}$ which implies an error of the order 10^{-2}. We insert this and (79) into (78):

$$\frac{c_6}{u_{\rm s}} \approx \frac{\rho_{\rm s}}{\rho}\,\frac{T}{c_P}\,\frac{\partial s_{\rm s}(T,P)}{\partial T}\,. \tag{80}$$

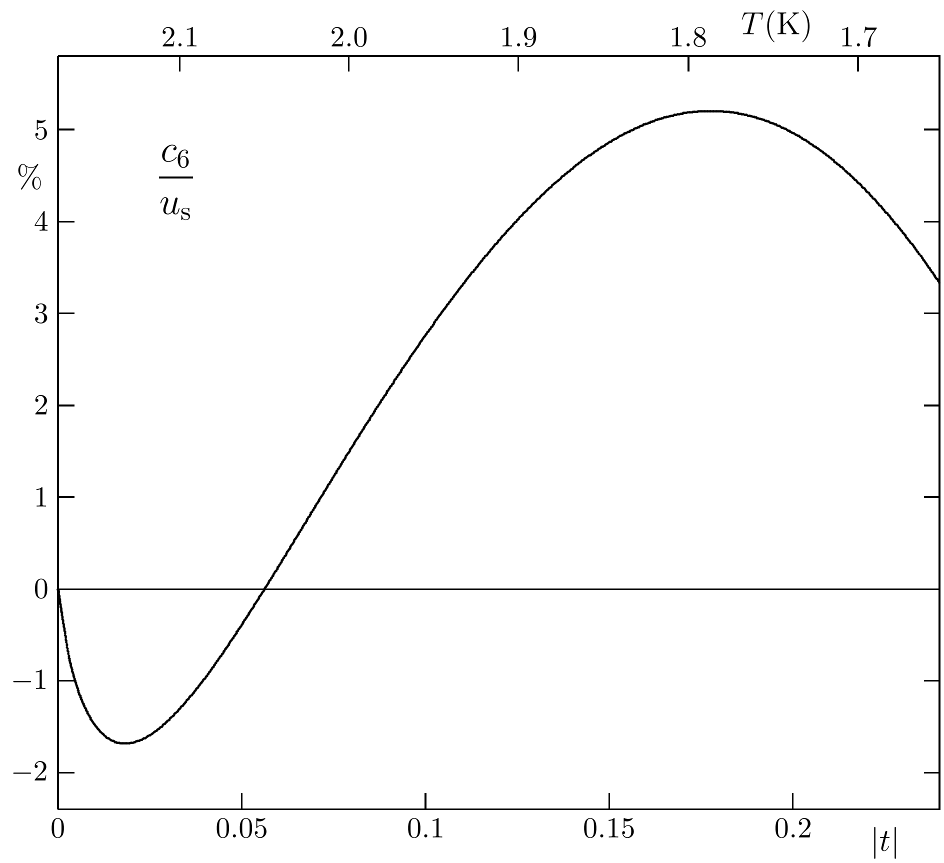

Figure 3: Velocity c_6 of the sixth sound in the ring experiment of Figure 2 as a function of the temperature T for saturated vapour pressure.

We use the prediction $s_{\mathrm{s}}(T, P)$ of Fig. 1 and the experimental ρ_{s}/ρ and c_P. Fig. 3 displays the resulting temperature dependence of c_6/u_{s}.

The shift velocity of the temperature variation (75) along the ring amounts up to a few percent of the supervelocity. Depending on the temperature range the shift is parallel or antiparallel to the persistent current (Fig. 3).

In an experiment, the temperature will be monitored at fixed points of the ring. At a given point the temperature amplitude oscillates with the frequency

$$\omega_{\mathrm{FP}} = \frac{c_6}{R}\, n\,. \tag{81}$$

For $s_{\mathrm{s}} = 0$ (or $u_{\mathrm{s}} = 0$) the considered mode becomes a static FP gradient along the ring. Therefore, we call (81) the 'fountain pressure' frequency.

As an example we insert the values $u_{\mathrm{s}} = 2\,\mathrm{cm/s}$ and $R = 2\,\mathrm{cm}$ of an actual experiment [16]. Using $c_6/u_{\mathrm{s}} \sim 3\%$ and $n = 1$ we obtain an oscillation period $2\pi/\omega_{\mathrm{FP}}$ of about three minutes.

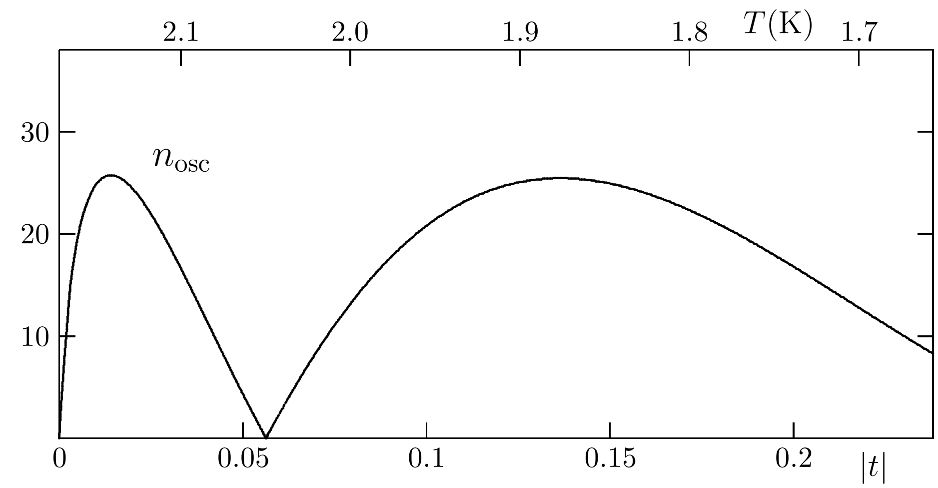

Figure 4: Number n_{osc} of observable fountain pressure oscillations for a ring experiment ($R = 2\,\mathrm{cm}$, $u_s = 2\,\mathrm{cm/s}$) as a function of the temperature at saturated vapour pressure.

Due to the damping (74) of the sixth sound the amplitude of the FP oscillation will be reduced by a factor e after

$$n_{\mathrm{osc}} = \frac{|\omega_{\mathrm{FP}}|}{2\pi\Gamma_{\mathrm{FP}}} = \frac{T\rho_{\mathrm{s}}}{2\pi m\kappa}\left|\frac{\partial s_{\mathrm{s}}(T,P)}{\partial T}\,u_{\mathrm{s}}\right|\frac{R}{n} \tag{82}$$

cycles of oscillation. We insert $n = 1$, $R = 2\,\mathrm{cm}$ and $u_{\mathrm{s}} = 2\,\mathrm{cm/s}$. We use the prediction $s_{\mathrm{s}}(T,P)$ of Fig. 1 and the experimental ρ_{s}. The heat conductivity $\kappa = \kappa(\mathrm{He})$ of He II is not well-known. For evaluating (82) we assume $\kappa = 0.05\,\mathrm{J/(m\,s\,K)}$; this value is cited in [3] as an upper limit for $T = 2.1\,\mathrm{K}$. The result is displayed in Fig. 4.

At the maximum of s_{s} in Fig. 1 the velocity c_6 and the number n_{osc} vanish. The exact position of this point is subject to the uncertainty in the predicted s_{s}. It is also slightly shifted by the approximation $c_{\mu,\mathrm{s}} \approx c_{P,\mathrm{s}}$.

The condition $n_{\mathrm{osc}} \gg 1$ for easy observability is fulfilled at most temperatures. Moreover, the number $n_{\mathrm{osc}} \propto |u_{\mathrm{s}}|R$ of observable oscillations could be increased by using a higher supervelocity or a larger ring. The sixth sound or, equivalently, the FP oscillations should be readily detectable.

The integration of eq. (80) yields

$$s_{\mathrm{s}}(T,P) \approx \int_{T_\lambda}^{T} dT'\,\frac{c_6/u_{\mathrm{s}}}{\rho_{\mathrm{s}}/\rho}\,\frac{c_P}{T'}\;. \tag{83}$$

The quantities c_6/u_{s}, ρ_{s}/ρ and c_P are functions of T' and P. All these quantities can be measured. For the evaluation of the integral they have to be measured in the temperature range from T_λ (where c_6/u_{s} and ρ_{s}/ρ vanish) to T and at fixed pressure. In this way the proposed experiment determines the superfluid entropy $s_{\mathrm{s}}(T,P)$.

If the sixth sound is not observed the considered experiment yields an upper limit for the superfluid entropy which is roughly given by

$$\left(\frac{S_{\rm s}}{S}\right)_{\rm upper\ limit} \approx \frac{1}{n_{\rm osc}} \left(\frac{S_{\rm s}}{S}\right)_{\rm predicted} \sim \frac{1\%}{n_{\rm osc}} \,. \tag{84}$$

In this way the present upper limit could be lowered by at least one order of magnitude.

5 Concluding remarks

A superfluid entropy of relative size $S_{\rm s}/S \sim 1\%$ is not excluded experimentally. We have presented a straight-forward generalization of the two-fluid equations for a non-vanishing superfluid entropy. The investigation of sound modes leads to a specific proposal by which a superfluid entropy of the considered size could be detected. A negative experiment would yield a new upper limit for the entropy content of the superfluid fraction.

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