Ergebnisse Aufgaben Math. I, II Chemie

1) a) $\sqrt{x+1}+\sqrt{x+6}=5 \Rightarrow \sqrt{x+6}^2=(5-\sqrt{x+1})^2 \Rightarrow \sqrt{x+1}=2 \Rightarrow x=3 \quad \sqrt{3+1}+\sqrt{3+6}=5$ ✓

b) $\frac{x-1}{x+1}<x \underset{x\neq -1}{\Leftrightarrow} \frac{x-1}{x+1}-x<0 \Leftrightarrow -\frac{x^2+1}{x+1}<0 \Leftrightarrow x>-1.$

c) $\left|\frac{x-2}{x+1}\right|<1 \underset{x\neq -1}{\Leftrightarrow} |x-2|<|x+1| \Leftrightarrow x>\frac{1}{2}$

2) Induktionsanfang $n=1$ $\prod_{k=1}^{1}\left(1+\frac{1}{k+1}\right)=1+\frac{1}{2}=\frac{3}{2}=\frac{1}{2}(1+2)$ ✓

Ind. Annahme $\prod_{k=1}^{n}\left(1+\frac{1}{k+1}\right)=\frac{1}{2}(n+2)$ für ein $n\geq 1$.

$n\to n+1 \Rightarrow \prod_{k=1}^{n+1}\left(1+\frac{1}{k+1}\right)=\left(\prod_{k=1}^{n}\left(1+\frac{1}{k+1}\right)\right)\left(1+\frac{1}{n+2}\right)\underset{\text{In.A.}}{=}\frac{1}{2}(n+2)\left(1+\frac{1}{n+2}\right)$

$=\frac{1}{2}(n+1+2)$ ✓

3) a) $\frac{z_2 z_3}{z_1}=\frac{1}{13}(-40+5i)$ b) $\alpha+i\beta=\sqrt{3+4i} \Rightarrow (\alpha+i\beta)^2=3+4i$

$\Rightarrow \alpha^2-\beta^2=3 \wedge 2\alpha\beta=4 \Rightarrow \beta=\frac{2}{\alpha} \Rightarrow \alpha^2-\frac{4}{\alpha^2}=3 \Rightarrow \alpha^2=-\frac{3}{2}+\sqrt{\frac{25}{4}}$

$\Rightarrow \alpha^2=1 \quad \alpha=\pm 1 \quad \beta=\pm 2 \Rightarrow \sqrt{3+4i}=\pm(1+2i)$

4) a) $z^2+(2+i)z-1+i=0 \Rightarrow z_{1,2}=-\frac{1}{2}(2+i)+\sqrt{\frac{1}{4}(2+i)^2+1-i}$

b) Mit $z=x+iy$: $|z+i|=\sqrt{x^2+(y+1)^2}=2x \Rightarrow x\geq 0$ und $x^2+(y-1)^2=4x^2$

$\Rightarrow x\geq 0 \wedge (y-1)^2=3x^2 \Rightarrow x\geq 0 \wedge y=1\pm\sqrt{3}x$. $z=x+(1\pm\sqrt{3}x)i$

5) a) $\left(\begin{array}{ccc|c}1&1&1&1\\1&1&-2&4\\2&1&-1&2\end{array}\right) \sim \left(\begin{array}{ccc|c}1&1&1&1\\0&0&-3&3\\0&-1&-3&0\end{array}\right) \Rightarrow (x,y,z)=(-1,3,-1)$

b) $D=\begin{vmatrix}1&1&1\\1&1&-2\\2&1&-1\end{vmatrix}=\begin{vmatrix}1&1&1\\0&0&-3\\1&0&-2\end{vmatrix}=-3 \quad x=-\frac{1}{3}\begin{vmatrix}1&1&1\\4&1&-2\\2&1&-1\end{vmatrix}=-\frac{1}{3}\begin{vmatrix}1&1&1\\3&0&-3\\1&0&-2\end{vmatrix}=-1$

$y=-\frac{1}{3}\begin{vmatrix}1&1&1\\1&4&-2\\2&2&-1\end{vmatrix}=-\frac{1}{3}\begin{vmatrix}1&1&1\\0&3&-3\\0&0&-3\end{vmatrix}=3 \quad z=-\frac{1}{3}\begin{vmatrix}1&1&1\\1&1&4\\2&1&2\end{vmatrix}=-\frac{1}{3}\begin{vmatrix}1&1&1\\0&0&3\\1&0&1\end{vmatrix}=-1$

6) $\det A=2\begin{vmatrix}1&1&-1&1\\1&-1&0&1\\1&1&1&2\\1&1&-1&-1\end{vmatrix}=2\begin{vmatrix}1&1&-1&1\\0&-2&1&0\\0&0&2&1\\0&0&0&-2\end{vmatrix}=16$

7) $\det A=\begin{vmatrix}1&-1&1\\1&0&2\\0&0&2\end{vmatrix}=2 \Rightarrow A^{-1}$ ex. $\quad A^{-1}=\frac{1}{2}\begin{pmatrix}-2&2&-2\\-3&2&-1\\1&0&1\end{pmatrix}$

8) a) $\sum_{j=0}^{\infty}\frac{(-3)^j}{j^2+1}x^j \quad a_j=\frac{(-3)^j}{j^2+1} \Rightarrow \lim_{j\to\infty}\left|\frac{a_j}{a_{j+1}}\right|=\lim_{j\to\infty}\frac{1}{3}\frac{(j+1)^2+1}{j^2+1}=\frac{1}{3}$

$\Rightarrow$ Konvergenzradius $r=\frac{1}{3}$

b) $\sum_{j=0}^{\infty}\frac{j^j}{(2j)!}(x+1)^j \quad a_j=\frac{j^j}{(2j)!}, \lim_{j\to\infty}\left|\frac{a_j}{a_{j+1}}\right|=\lim_{j\to\infty}\frac{j^j}{(j+1)^{j+1}}\frac{(2j+2)!}{(2j)!}$

$=\lim_{j\to\infty}\frac{j^j(2j+2)(2j+1)}{(j+1)^j(j+1)}=\lim_{j\to\infty}\underbrace{\left(\frac{1}{1+\frac{1}{j}}\right)^j}_{\to\frac{1}{e}}\cdot 2(2j+1)=+\infty$

Reihe beständig konvergent

9) $P(x) = x^4 + x^3 - 6x^2 - 4x + 8$

$$\begin{array}{r|rrrrr} & 1 & 1 & -6 & -4 & 8 \\ 1 & & 1 & 2 & -4 & -8 \\ \hline & 1 & 2 & -4 & -8 & \lfloor 0 \end{array} \quad P(1) = 0$$

$$\begin{array}{r|rrrrr} & 1 & 1 & -6 & -4 & 8 \\ -1 & & -1 & 0 & 6 & -2 \\ \hline & 1 & 0 & -6 & 2 & \lfloor 6 = P(-1) \end{array}$$

$$\begin{array}{r|rrrrr} & 1 & 1 & -6 & -4 & 8 \\ 2 & & 2 & 6 & 0 & -8 \\ \hline & 1 & 3 & 0 & -4 & \lfloor 0 \end{array} \quad P(2) = 6$$

$$\begin{array}{r|rrrrr} & 1 & 1 & -6 & -4 & 8 \\ -2 & & -2 & 2 & 8 & -8 \\ \hline & 1 & -1 & -4 & 4 & \lfloor 0 \end{array} \quad P(-2) = 0$$

$$\begin{array}{r|rrrrr} & 1 & 1 & -6 & -4 & 8 \\ 3 & & 3 & 12 & 18 & 42 \\ \hline & 1 & 4 & 6 & 14 & \lfloor 50 \end{array} \quad P(3) = 50$$

b)

$$\begin{array}{r|rrrrr} & 1 & 1 & -6 & -4 & 8 \\ 2 & & 2 & 6 & 0 & -8 \\ \hline & 1 & 3 & 0 & -4 & 0 \\ -2 & & -2 & -2 & 4 & \\ \hline & 1 & 1 & -2 & \boxed{0} & \\ 1 & & 1 & 2 & & \\ \hline & 1 & 2 & \lfloor 0 & & \end{array}$$

$\Rightarrow P(x) = (x-2)^2(x+2)(x-1)$

10) a) $\lim\limits_{x \to 0} \dfrac{\cos(3x) - 1}{\sin^2 x} \overset{(\frac{0}{0})}{=} \lim\limits_{x \to 0} \dfrac{-3\sin(3x)}{2\sin x \cos x} \overset{\frac{0}{0}}{=} \lim\limits_{x \to 0} \dfrac{-9\cos(3x)}{2(\cos^2 x - \sin^2 x)}$

Regel von de l'Hospital

$= -\frac{9}{2}$

b) $\lim\limits_{x \to +\infty} \dfrac{e^x}{\sinh x}$ Hier ist die Regel von de l'Hosp. anwendbar führt aber zu keinem Resultat

$\sinh x = \frac{1}{2}(e^x - e^{-x}) \Rightarrow \lim\limits_{x \to +\infty} \dfrac{e^x}{\sinh x} = \lim\limits_{x \to +\infty} 2\dfrac{e^x}{e^x - e^{-x}} = 2\lim\limits_{x \to +\infty} \dfrac{1}{1 - e^{-2x}}$

$\Rightarrow$ ererement 2 da $\lim\limits_{x \to +\infty} e^{-2x} = 0.$

c) $\lim\limits_{x \to 1} (x-1)\cot(\pi x) = \lim\limits_{x \to 1} \dfrac{(x-1)\cos(\pi x)}{\sin(\pi x)} \; (\frac{0}{0})$

$= \lim\limits_{x \to 1} \dfrac{\cos(\pi x) + (x-1)\pi\sin(\pi x)}{\pi\cos(\pi x)} = \dfrac{1}{\pi}$

11) $f(x) = \ln(\frac{x}{3}) = \ln x - \ln 3 \quad f'(x) = \frac{1}{x}, \; f''(x) = -\frac{1}{x^2}, \; f'''(x) = \frac{2}{x^3}$

$P_2(x) = f(3) + f'(3)(x-3) + f''(3)\frac{1}{2}(x-3)^2 \qquad f(3) = \ln 1 = 0 \quad f'(3) = \frac{1}{3}$

$f''(3) = -\frac{1}{9} \Rightarrow P_2(x) = \frac{1}{3}(x-3) - \frac{1}{18}(x-3)^2$

$\Delta(x) = |f(x) - P_2(x)| = \left|\frac{(x-3)^3}{3!} f'''(3 + \delta(x-3))\right|$ mit $0 < \delta < 1$.

$\Rightarrow \Delta(x) = \dfrac{|x-3|^3}{6} \dfrac{2}{|3 + \delta(x-3)|^3}, \quad |x-3| \le \frac{1}{2} \quad 0 < \delta < 1 \Rightarrow |3 + \delta(x-3)| \ge \frac{5}{2}$

$\Rightarrow \Delta(x) \le \frac{1}{3}\left(\frac{1}{2}\right)^3 \dfrac{1}{(\frac{5}{2})^3} = \dfrac{1}{3 \cdot 5^3} = \dfrac{1}{375} < 3 \cdot 10^{-3}$

12 a) $\int (x^3 + 4\cosh(2x))\,dx = \frac{1}{4}x^4 + 2\sinh(2x) + C$

b) $J = \int x e^{-2x}\,dx$ Partielle Integration $f(x) = x \quad g'(x) = e^{-2x}$

$f' = 1 \quad g(x) = -\frac{1}{2}e^{-2x}$

$J = f \cdot g - \int g f'\,dx = -\frac{1}{2}x e^{-2x} + \int \frac{1}{2}e^{-2x}\,dx = -\frac{1}{2}e^{-2x}(x + \frac{1}{2}) + C$

c) $J = \int 1 \cdot \ln(1-x^2)\,dx \quad = x\ln(1-x^2) - \int x \cdot \dfrac{-2x}{1-x^2}\,dx$

$f' = 1 \quad g = \ln(1-x^2)$

$J = x\ln(1-x^2) + 2\int \dfrac{x^2 - 1 + 1}{1-x^2}\,dx = x\ln(1-x^2) + 2\int\left(1 + \dfrac{1}{x^2-1}\right)dx$

$\Rightarrow J = x\ln(1-x^2) + 2x + \int\left(\frac{1}{x+1} - \frac{1}{x-1}\right)dx = x(\ln(1-x^2) + 2) + \ln\left|\frac{x+1}{x-1}\right| + C$

13 a) $J = \int x^3 \cos(x^2)\,dx = \frac{1}{2}\int x^2 \cos(x^2)\, 2x\,dx = \frac{1}{2}\int u \cos u\,du$

Subst $u = x^2 \quad du = 2x\,dx$

Part. Integration

$\int u \cos u\,du = u \sin u - \int \sin u\,du = u \sin u + \cos u + c$

$f(u) = u \quad g'(u) = \cos u$ Rücksubst. $u = x^2$

$f' = 1 \quad g(u) = \sin u$

$J = \frac{1}{2}\left(x^2 \sin(x^2) + \cos(x^2)\right) + c$

b) wegen $x^3 + x^2 + x + 1 = (x+1)(x^2+1)$ ist $\frac{x+1}{x^3+x^2+x+1} = \frac{1}{x^2+1}$

$\Rightarrow \int \frac{x+1}{x^3+x^2+x+1}\,dx = \int \frac{1}{1+x^2}\,dx = \arctan x + c$

c) $J = \int \frac{1}{x^3 - 2x^2 + x - 2}\,dx$ Nullstellen des Nenners $x_1 = 2$

$$\begin{array}{c|cccc} & 1 & -2 & 1 & -2 \\ 2 & & 2 & 0 & 2 \\ \hline & 1 & 0 & 1 & 0 \end{array}$$

$\Rightarrow \frac{1}{x^3 - 2x^2 + x - 2} = \frac{1}{(x-2)(x^2+1)} = \frac{a}{x-2} + \frac{bx+c}{x^2+1} \Rightarrow 1 = a(x^2+1) + (bx+c)(x-2)$

$\Rightarrow 1 = x^2\underbrace{(a+b)}_{=0} + x\underbrace{(c-2b)}_{=0} + \underbrace{a-2c}_{=1} \Rightarrow a - 2c = 1,\ a + b = 0,\ c - 2b = 0$

$\Rightarrow a = \frac{1}{5} \quad b = -\frac{1}{5} \quad c = -\frac{2}{5}$

$\Rightarrow J = \frac{1}{5}\left[\int \frac{1}{x-2}\,dx - \frac{1}{2}\int \frac{2x}{x^2+1}\,dx - 2\int \frac{dx}{x^2+1}\right]$

$\Rightarrow J = \frac{1}{5}\left[\ln|x-2| - \frac{1}{2}\ln(x^2+1) - 2\arctan x\right] + c$

14 a) wegen $x^3 + x^2 - 2 = (x-1)(x^2+2x+2) \Rightarrow \frac{x-1}{x^3+x^2-2} = \frac{1}{x^2+2x+2}$

$J = \int \frac{1}{x^2+2x+2}\,dx = \int \frac{1}{(x+1)^2+1}\,dx = \int \frac{1}{u^2+1}\,du = \arctan u + c$ $\quad u = x+1 \quad dx = du$

$u = x+1$

$\Rightarrow \int \frac{1}{x^2+2x+2}\,dx = \arctan(x+1) + c$ (die Stelle $x = 1$ ist hebbar)

und damit $\int_1^2 \frac{x-1}{x^3+x^2-2}\,dx = \arctan(x+1)\Big|_1^2 = \arctan 3 - \arctan 2$

b) $\int_{-1}^{2} \frac{x}{x^4+5x+1}\,dx$ existiert nicht, da bei $x = -1$ der Nenner gleich Null wird. (Polstelle).

15) a) Es ist $\vec{a} = (2,-1,2)$, $\vec{b} = (2,1,-1)$ $\|\vec{a}\| = \sqrt{9} = 3$ $\|\vec{b}\| = \sqrt{6}$ $\Rightarrow \vec{a}_0 = \frac{1}{\|\vec{a}\|}\vec{a}$

$\Rightarrow \vec{a}_0 = \frac{1}{3}(2,-1,2)$, $\vec{b}_0 = \frac{1}{\sqrt{6}}(2,1,-1)$

b) Volumen: $V = |[\vec{a},\vec{b},\vec{d}]| = \left|\begin{vmatrix} 2 & -1 & 2 \\ 2 & 1 & -1 \\ 2 & 4 & 1 \end{vmatrix}\right| = \left|\begin{vmatrix} 2 & -1 & 2 \\ 0 & 2 & -3 \\ 0 & 5 & -1 \end{vmatrix}\right| = \left|2\begin{vmatrix} 2 & -3 \\ 5 & -1 \end{vmatrix}\right|$

$\Rightarrow V = 26$

Oberfläche ist $F = 2\left[\|\vec{a}\times\vec{b}\| + \|\vec{a}\times\vec{d}\| + \|\vec{b}\times\vec{d}\|\right]$

wegen $\vec{a}\times\vec{b} = (-1,6,4)$, $\vec{a}\times\vec{d} = (-3,6,6)$, $\vec{b}\times\vec{d} = (1,0,2)$

$\Rightarrow F = 2\left[\sqrt{53} + 9 + \sqrt{5}\right]$

c) Die Projektion von $\vec{b}$ auf $\vec{a}$ ist: $b_1 = \frac{\vec{a}\,\vec{b}}{\|\vec{a}\|^2}\vec{a}$

$\vec{a}\vec{b} = (2,-1,2)(2,1,-1) = 1 \Rightarrow \vec{b}_1 = \frac{1}{9}\vec{a} = \frac{1}{9}(2,-1,2)$

$b_2 = \vec{b} - b_1 = (2,1,-1) - \frac{1}{9}(2,-1,2) = \frac{1}{9}(16,10,-10)$

16) a) $V = |[\vec{a},\vec{b},\vec{c}]| = \left|\det\begin{pmatrix}1&-1&1\\t&2&2-t\\2&2&-1\end{pmatrix}\right| = \left|\begin{vmatrix}0&-1&0\\t+2&2&4-t\\4&2&1\end{vmatrix}\right|$

$= \left|\begin{vmatrix}t+2&4-t\\4&1\end{vmatrix}\right| = |5t-14| = 18 \Rightarrow 5t-14=18 \vee 14-5t=18$

$\Rightarrow 5t = 32 \vee 5t=-4 \Rightarrow t_1 = \frac{32}{5}, \; t_2 = -\frac{4}{5}$

b) Projektion von $\vec{b}$ auf $\vec{d}$ ist $\vec{b}_p = \frac{\vec{b}\cdot\vec{d}}{\|\vec{d}\|^2}\vec{d}$

$\vec{d}\vec{b} = (2,2,-1)(t,2,2-t) = 3t$, also wegen $\|\vec{d}\|^2=9$

$\vec{b}_p = \frac{t}{3}(2,2,-1) \Rightarrow \|\vec{b}_p\| = |t| = 3 \Leftrightarrow t = \pm 3.$

c) $a,b,\vec{d}$ linear abhängig $\Leftrightarrow [a,b,\vec{d}]=0$

$\Rightarrow \begin{vmatrix}1&-1&1\\t&2&2-t\\2&2&-1\end{vmatrix} = \begin{vmatrix}1&0&0\\t&t+2&2-2t\\2&4&-3\end{vmatrix} = \begin{vmatrix}t+2&2-2t\\4&-3\end{vmatrix} = 5t-14=0$

$\Rightarrow t = \frac{14}{5} = 2{,}8$

17) Eigenwerte von A : $\det(A-\lambda E) = \begin{vmatrix}2-\lambda&4&-3\\4&2-\lambda&-3\\-3&-3&9-\lambda\end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix}-2-\lambda&2-\lambda&0\\4&2-\lambda&-3\\-3&-3&9-\lambda\end{vmatrix} = \begin{vmatrix}-2-\lambda&0&0\\4&6-\lambda&-3\\-3&-6&9-\lambda\end{vmatrix} = -(2+\lambda)\begin{vmatrix}6-\lambda&-3\\-6&9-\lambda\end{vmatrix}$

$\Rightarrow (\lambda+2)(\lambda^2-15\lambda+36)=0 \quad \lambda_1=-2 \quad \lambda_{2,3} = \frac{15}{2} \pm \sqrt{\left(\frac{15}{2}\right)^2-36}$

$\lambda_2 = 3, \; \lambda_3 = 12$

Eigenvektorräume zu

λ_1: $(A+2E)\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} = \vec{0} \Rightarrow \left(\begin{array}{ccc|c}4&4&-3&0\\4&4&-3&0\\-3&-3&7&0\end{array}\right) \Rightarrow a_3=0 \quad a_2=-a_1$

$\Rightarrow \vec{a} = t(1,-1,0) = \langle(1,-1,0)\rangle$

λ_2: $(A-3E)\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} = \vec{0} \Rightarrow \left(\begin{array}{ccc|c}-1&4&-3&0\\4&-1&-3&0\\-3&-3&6&0\end{array}\right) \sim \left(\begin{array}{ccc|c}-1&4&-3&0\\5&-5&0&0\\-5&5&0&0\end{array}\right) \Rightarrow \vec{a} = t(1,1,1)$

$= \langle(1,1,1)\rangle$

λ_3: $(A-\;E)\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} = \vec{0}$

$\Rightarrow \left(\begin{array}{ccc|c}-10&4&-3&0\\4&-10&-3&0\\-3&-3&-3&0\end{array}\right) \sim \left(\begin{array}{ccc|c}-7&7&0&0\\7&-7&0&0\\-3&-3&-3&0\end{array}\right) \sim \left(\begin{array}{ccc|c}0&0&0&0\\1&-1&0&0\\1&1&1&0\end{array}\right) \Rightarrow \vec{a} = t(1,1,-2)$

$= \langle(1,1,-2)\rangle$

18 a) $\frac{\partial}{\partial x} f = \frac{\partial}{\partial x} x\cos(x-y) = \cos(x-y) - x\sin(x-y) \quad \frac{\partial}{\partial y} f = x\sin(x-y)$

$\frac{\partial^2}{\partial x^2} f = -2\sin(x-y) - x\cos(x-y) \quad \frac{\partial^2}{\partial x\partial y} f = \frac{\partial^2}{\partial y\partial x} f = -x\cos(x-y)$

$\frac{\partial^2}{\partial y^2} f = -x\cos(x-y)$

b) $\frac{\partial}{\partial x} g = \frac{2x}{x^2+y^2} \quad \frac{\partial}{\partial y} g = \frac{2y}{x^2+y^2}, \quad \frac{\partial^2}{\partial x^2} g = 2\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$

$\frac{\partial^2}{\partial x^2} g = 2\frac{y^2-x^2}{(x^2+y^2)^2}, \quad \frac{\partial^2}{\partial y^2} g = \frac{x^2-y^2}{(x^2+y^2)^2} \quad \frac{\partial^2}{\partial x\partial y} g = \frac{\partial^2}{\partial y\partial x} g = -\frac{4xy}{(x^2+y^2)^2}$

c) $\frac{\partial}{\partial x} h = 2\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) = -\frac{2y}{x^2+y^2} \quad \frac{\partial}{\partial y} h = 2\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x} = \frac{2x}{x^2+y^2}$

$\frac{\partial^2}{\partial x^2} h = \frac{4xy}{(x^2+y^2)^2} \quad \frac{\partial^2}{\partial y^2} h = -\frac{4xy}{(x^2+y^2)^2} \quad \frac{\partial^2}{\partial x\partial y} h = \frac{\partial^2}{\partial y\partial x} h = 2\frac{y^2-x^2}{(x^2+y^2)^2}$

21) a) $(x+1)y' = xy^2 + x - y^2 - 1 = (x-1)(y^2+1)$ trennbare Dgl.

$\Rightarrow (x+1)\frac{dy}{dx} = (y^2+1)(x-1) \Rightarrow \frac{1}{y^2+1}dy = \frac{x-1}{x+1}dx$

$\Rightarrow \int \frac{1}{y^2+1}dy = \int \frac{x-1}{x+1}dx \Rightarrow \arctan y = \int (1 - \frac{2}{x+1})dx$

$\Rightarrow \arctan y = x - 2\ln|x+1| + C \Rightarrow y = \tan(x - 2\ln|x+1| + c)$

b) $xy' - y' = x$ Lineare Dgl. 1. Ordnung, Normierung:

$y' - \frac{1}{x}y' = 1 \Rightarrow y_A = e^{-\int -\frac{1}{x}dx}\left[\int 1 \cdot e^{-\int \frac{1}{x}dx} dx + c\right]$

$\Rightarrow y_A = x\left(\int \frac{1}{x}dx + c\right) = x(\ln|x| + c)$

22) a) $y' \cdot \cos x + y \sin x = \cos^2 x$ Lineare 1. Ordnung, Normierung

$y' + \frac{\sin x}{\cos x}y = \cos x \Rightarrow y_A = e^{-\int \frac{\sin x}{\cos x}dx}\left[\int \cos x\, e^{\int \frac{\sin x}{\cos x}dx} dx + C\right]$

$\Rightarrow y = e^{\ln \cos x}\left[\int \cos x\, e^{-\ln \cos x} dx + c\right]$

$y = \cos x\left[\int \frac{\cos x}{\cos x}dx + C\right] \Rightarrow y = \cos x(x + C)$

$y(0) = C = 1 \Rightarrow y = (x+1)\cos x$

b) $y' = \frac{dy}{dx} = -\frac{2xy}{x^2+y^2} \Rightarrow 2xy\,dx + (x^2+y^2)dy = 0$

Mit $P = 2xy$, $Q = x^2+y^2$ gilt $P_y = 2x = Q_x$

$\Rightarrow$ Dgl. ist exakt, Potentialfunktion $f_x = P \wedge f_y = Q$

$f(x,y) = \int P\,dx + \varphi(y) = \int 2xy\,dx + \varphi(y) = yx^2 + \varphi(y)$

$\frac{\partial}{\partial y}f = x^2 + \varphi'(y) = Q = x^2 + y^2 \Rightarrow \varphi'(y) = y^2 \Rightarrow \varphi(y) = \frac{1}{3}y^3$

$\Rightarrow f(x,y) = yx^2 + \frac{1}{3}y^3 \Rightarrow yx^2 + \frac{1}{3}y^3 = C$ ist die allgemeine Lösung $y(1) = 1 \Rightarrow C = \frac{4}{3} \Rightarrow yx^2 + \frac{1}{3}y^3 = \frac{4}{3}$ ist die Lösung des AWP's.

23 a) $y''' - y'' - y' + y = x$ inhomogene Lin. Dgl. 3. Ordnung mit konstanten Koeff. Lösungsweg

(I) Lösung der hom. Dgl. $y''' - y'' - y' + y = 0$ mit Ansatz

$y = e^{\lambda x} \Rightarrow$ Charakt.Gl. $\lambda^3 - \lambda^2 - \lambda + 1 = 0 \Rightarrow \lambda_1 = 1$, $\lambda_2 = \lambda_1 = 1$

$\lambda_3 = -1 \Rightarrow y_h = (c_1 + c_2 x)e^x + c_3 e^{-x}$ (da 1 2-fache Nullst.)

II Inhomogene Dgl. $s(x) = x$ (Polynom 1. Grades)

Ansatz $y_p = \alpha x + \beta \Rightarrow y_p' = \alpha$, $y_p'' = y_p''' = 0$

$\Rightarrow -\alpha + \alpha x + \beta = x \Rightarrow x(\alpha - 1) + \beta - \alpha = 0 \Rightarrow \alpha = 1$, $\beta = -1$

$y_p = x - 1$, $y_A = y_h + y_p = (c_1 + c_2 x)e^x + c_3 e^{-x} + x - 1$

b) $y''' + 5y'' + 3y' - 9y = e^x$

I hom. Dgl. $y''' + 5y'' + 3y' - 9y = 0$, Ansatz $y = e^{\lambda x} \Rightarrow$

Char. Gl. $\lambda^3 + 5\lambda^2 + 3\lambda - 9 = 0 \Rightarrow \lambda_1 = 1$,
$$\begin{array}{r|rrrr} & 1 & 5 & 3 & -9 \\ 1 & & 1 & 6 & 9 \\ \hline & 1 & 6 & 9 & 0 \end{array}$$

$\Rightarrow \lambda^2 + 6\lambda + 9 = 0 \quad \lambda_{2,3} = -3$

$\Rightarrow y_h = c_1 e^x + (c_2 + c_3 x) e^{-3x}$

II Inhom. Dgl. $s(x) = e^{1 \cdot x}$. Da 1 einfache Nullst der char. Gl. Ansatz für $y_p = \alpha x e^x \Rightarrow y_p' = \alpha(x+1)e^x$

$y_p'' = \alpha(x+2)e^x$, $y_p''' = \alpha(x+3)e^x \Rightarrow$

$\alpha e^x [x + 3 + 5(x+2) + 3(x+1) - 9x] = e^x$

$\Rightarrow 15\alpha = 1 \Rightarrow \alpha = \frac{1}{15}$, $y_p = \frac{1}{15} x e^x$, $y_A = y_h + y_p$

$\Rightarrow y = c_1 e^x + (c_2 + c_3 x) e^{-3x} + \frac{1}{15} x e^x$

24) $y'' + 2y' + 2y = e^x \quad y(0) = y'(0) = 0$

I hom. Dgl. $y'' + 2y' + 2y = 0$, Ansatz $y = e^{\lambda x} \Rightarrow$

Char. Gl. $\lambda^2 + 2\lambda + 2 = 0 \Rightarrow \lambda_{1,2} = -1 \pm \sqrt{-1} = -1 \pm i$

$\Rightarrow y_h = e^{-x}(c_1 \cos x + c_2 \sin x)$

II Inhom. Dgl. $s(x) = e^{1x}$ (1 keine Nullst der Ch. Gl.)

$\Rightarrow$ Ansatz $y_p = \alpha e^x$, $y_p' = y_p'' = \alpha e^x$

$\Rightarrow 5\alpha e^x = e^x \Rightarrow \alpha = \frac{1}{5} \Rightarrow y_p = \frac{1}{5} e^x$,

$y_A = y_h + y_p' = e^{-x}(c_1 \cos x + c_2 \sin x) + \frac{1}{5} e^x$

$y' = e^{-x}[(c_2 - c_1)\cos x - (c_1 + c_2)\sin x) + \frac{1}{5} e^x$

Anfangswerte

$y(0) = c_1 + \frac{1}{5} = 0 \ \wedge \ y'(0) = c_2 - c_1 + \frac{1}{5} = 0 \Rightarrow c_1 = -\frac{1}{5}$

$c_2 = -\frac{2}{5} \Rightarrow y = -\frac{1}{5} e^{-x}[\cos x + 2\sin x] + \frac{1}{5} e^x$

25) $\vec{y}' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^x$ inhomogenes System 1. Ordn.

I Lösung des hom. Systems $\vec{y}' = \begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix} \vec{y}$, Ansatz $\vec{y} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda x}$

$\Rightarrow$ Eigenwertproblem $\det \begin{pmatrix} 1-\lambda & 4 \\ 1 & -2-\lambda \end{pmatrix} = \lambda^2 + \lambda - 6 = 0$

$\lambda_{1,2} = -\frac{1}{2} \pm \sqrt{6 + \frac{1}{4}} = -\frac{1}{2} \pm \frac{5}{2} \quad \lambda_1 = 2,\ \lambda_2 = -3$

Eigenräume: E_1 $\begin{array}{rr|r} -1 & 4 & 0 \\ 1 & -4 & 0 \end{array} \Rightarrow a_1 - 4a_2 = 0 \quad E_1 = \{t(4,1)\}$

$E_2 \quad \lambda_2 = -3 \quad \begin{array}{rr|r} 4 & 4 & 0 \\ 1 & 1 & 0 \end{array} \quad E_2 = \{t(1,-1)\}$

$\Rightarrow \vec{y}_h = c_1 \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{2x} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3x}$

Inhom. System

$\vec{y}_A = \vec{y}_h + \vec{y}_p$, Da mit $\vec{y}\,' = A\vec{y} + \vec{b}$ $\vec{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^x$

und 1 kein Eigenwert : Ansatz $\vec{y}_p = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} e^x$

$\Rightarrow \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} e^x = \begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} e^x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^x \Rightarrow \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 + 4\alpha_2 \\ \alpha_1 - 2\alpha_2 + 1 \end{pmatrix}$

$\Rightarrow 4\alpha_2 = 0 \;, \alpha_1 + 1 = 0 \Rightarrow \alpha_1 = -1 \;, \alpha_2 = 0$

$\Rightarrow y_p = \begin{pmatrix} -1 \\ 0 \end{pmatrix} e^x \Rightarrow \vec{y}_A = c_1 \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{2x} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3x} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^x$