7. Aufgabenblatt Math. I ET WS 13/14

1 a) $f(x) = x\sin x + e^x$ $\quad f'(x) = \sin x + x\cos x + e^x$ $\quad D_f = D_{f'} = \mathbb{R}$

b) $f(x) = \ln\left((x^3e^x)^{\frac12}\cosh x\right) = \frac12 \ln(x^3e^x) + \ln\cosh x = \frac32\ln x + \frac12 x + \ln(\cosh x)$

$D_f = \mathbb{R}_+$ $\quad f'(x) = \frac32\frac1x + \frac12 + \tanh x$ $\quad D_{f'} = D_f$

c) $f(x) = 4^{\cos x} = e^{\ln 4\cdot\cos x} \Rightarrow f'(x) = -e^{\ln 4\cos x}\ln 4\sin x = -4^{\cos x}\ln 4\cdot\sin x$

$D_f = D_f = \mathbb{R}$

d) $f'(x) = \left(\cos(\arcsin\sqrt{1-x^2})\right)' = \sin(\arcsin\sqrt{1-x^2})\cdot\frac{1}{\sqrt{1-\sqrt{1-x^2}^2}}\frac{-x}{\sqrt{1-x^2}}$

$= \sqrt{1-x^2}\frac{1}{\sqrt{x^2}}\cdot\frac{-x}{\sqrt{1-x^2}} = -\frac{x}{|x|} = -\operatorname{sign} x$ $\quad D_f = [-1,1]$ $\quad D_{f'} = (-1,1)\setminus\{0\}$

e) $f' = \frac{-1}{\sqrt{1-\sin^2 x}}\cos x = \frac{-\cos x}{\sqrt{\cos^2 x}} = -\frac{\cos x}{|\cos x|} = -\operatorname{sign}(\cos x)$ $\quad D_f = \mathbb{R}$

$D_{f'} = \mathbb{R}\setminus\{(k+\frac12)\pi,\ k\in\mathbb{Z}\}$

f) $f(x) = x^{\frac{1}{\ln x}} = (e^{\ln x})^{\frac{1}{\ln x}} = e^1 = e$ $\quad D_f = \mathbb{R}^+\setminus\{1\}$ ≙

$f'(x) = 0$ $\quad D_{f'} = D_f$

g) $f(x) = \cosh\ln x^{\frac32} = \frac12\left(e^{\ln x^{\frac32}} + e^{-\ln x^{\frac32}}\right) = \frac12\left(x^{\frac32}+x^{-\frac32}\right)$

$f'(x) = \frac34\left(x^{\frac12} - x^{-\frac52}\right)$ $\quad D_f = \mathbb{R}_+ = D_{f'}$

2) Mit $f(x) = \sinh x - x$, gilt $f(0)=0$

$f'(x) = \cosh x - 1 > 0$ für $x>0$ und $f'(0)=0$, damit ist

$f(x)$ streng monoton wachsend für $x\ge 0$, $\Rightarrow$

$f(x) > f(0) = 0$ für $x>0 \Rightarrow f(x) = \sinh x - x \ge 0 \ \forall x\ge 0$

$\Rightarrow$ Beh.

3 a) $\lim_{x\to0}\frac{\sin^2 x}{1-\cosh x} \overset{(\frac00)}{=} \lim_{x\to0}\frac{2\sin x\cos x}{-\sinh x} = \lim_{x\to0}\frac{2(\cos^2x-\sin^2x)}{-\cosh x} = -2$

(2× Regel von de l'Hospital)

b) Da $0\le\sin^2 x\le 1$ gilt und $\lim_{x\to+\infty}\cosh x = +\infty \Rightarrow$

$\left|\frac{\sin^2 x}{1-\cosh x}\right| \le \frac{1}{\cosh x - 1}$ für $x>0 \Rightarrow \lim_{x\to+\infty}\frac{\sin^2x}{1-\cosh x} = 0$

c) $\lim_{x\to+\infty} e^{-x}\sinh x = \lim_{x\to+\infty} e^{-x}\frac12(e^x-e^{-x}) = \lim_{x\to+\infty}\left(\frac12 - \frac12 e^{-2x}\right) = \frac12$

d) $\Rightarrow e^{\lim_{x\to\frac12-}\frac{\ln\cos(\pi x)}{\ln(1-2x)}} \overset{(\frac{\infty}{\infty})}{=} e^{\lim_{x\to\frac12-}\frac{-\pi\sin(\pi x)}{\cos(\pi x)}/\frac{-2}{1-2x}} = e^{\frac{\pi}{2}\lim_{x\to\frac12-}\frac{1-2x}{\cos(\pi x)}}$

$= e^{\frac\pi2\lim_{x\to\frac12}\frac{-2}{-\pi\sin(\pi x)}} = e$

e) $\lim_{x\to0}\frac1x\left(\coth x - \frac{1}{\sinh x}\right) = \lim_{x\to0}\frac1x\left(\frac{\cosh x}{\sinh x} - \frac{1}{\sinh x}\right) = \lim_{x\to0}\frac{\cosh x - 1}{x\sinh x} \overset{(\frac00)}{=}$

$\lim_{x\to0}\frac{\sinh x}{\sinh x + x\cosh x} \overset{(\frac00)}{=} \lim_{x\to0}\frac{\cosh x}{2\cosh x + x\sinh x} = \frac12$

f) $\lim\limits_{x\to 0}(\cos x)^{\cot x} = e^{\lim\limits_{x\to 0}(\ln\cos x)\frac{1}{\tan x}} = e^{\lim\limits_{x\to 0}\frac{\ln\cos x}{\tan x}}$ $(\frac{0}{0})$

$= e^{\lim\limits_{x\to 0}\frac{-\sin x}{\cos x(1+\tan^2 x)}} = e^0 = 1$

g) $\lim\limits_{x\to+\infty}(\cosh x)^{\frac{1}{x}} = e^{\lim\limits_{x\to\infty}\frac{\ln\cosh x}{x}} = e^{\lim\limits_{x\to+\infty}\frac{\tanh x}{1}} = e$,

da $\lim\limits_{x\to+\infty}\tanh x = 1$.

4) Offenbar schneiden sich die Kurven bei $x_1=0$, $x_2=\frac{\pi}{2}$

Da $4\left(\frac{x}{\pi}\right)^2$ in $[0,\frac{\pi}{2}]$ und da mit $f(x)=4\frac{x^2}{\pi^2}$, $g(x)=\sin x$

gilt $f''(x)=\frac{8}{\pi^2}>0$ $g''(x)=-\sin x\le 0$ auf $[0,\frac{\pi}{2}]$ ⇒ f konvex, g konkav

⇒ kein weiterer Schnittpunkt in $(0,\frac{\pi}{2})$. Außerhalb des Intervalls ist stets $f(x)\neq g(x)$.

$f'(x) = \frac{8x}{\pi^2}$ $g'(x)=\cos x$ ⇒ $f'(0)=0$, $g'(0)=1$ $f'(\frac{\pi}{2})=\frac{4}{\pi}$, $g'(\frac{\pi}{2})=0$

Schnittwinkel: bei $(0,0)$ $\tan\alpha = \frac{0-1}{1} = -1$ ⇒ $\alpha = -\frac{\pi}{4}$

bei $(\frac{\pi}{2},1)$ $\tan\alpha = \frac{\frac{4}{\pi}-0}{1} = \frac{4}{\pi}$ ⇒ $\alpha = \arctan\frac{4}{\pi}$

5) Mit $f(x)=\arctan x$ gilt $f(1)=\frac{\pi}{4}$ $f'(x)=\frac{1}{1+x^2}$

Also folgt aus dem Mittelwertsatz der Diff.-Rechn.

$\frac{f(x)-f(1)}{x-1} = f'(\xi)$ mit $|\xi-1|<|x-1|$

⇒ $\frac{|\arctan x-\frac{\pi}{4}|}{|x-1|} = |f'(\xi)| = \frac{1}{1+\xi^2}$ $|\xi-1|<|x-1|<\frac{1}{2}$

⇒ $\frac{1}{2}<\xi<\frac{3}{2}$ ⇒ $\frac{1}{1+\xi^2} \le \frac{1}{1+\frac{1}{4}} = \frac{4}{5}$ ⇒ $|\arctan x-\frac{\pi}{4}| \le \frac{4}{5}|x-1|$

6 a) $z^4 = -4-4\sqrt{3}i$ ⇒ $z=\sqrt[4]{-4-4\sqrt{3}i}$ $|-4-4\sqrt{3}i| = 8$

$\phi = \arg(-4-4\sqrt{3}i) = \arctan\sqrt{3}+\pi = \frac{4}{3}\pi$ ⇒ $z_k = \sqrt[4]{8}\,e^{i\frac{\pi}{3}}\,\omega_{4,k}$ $k=0,1,2,3$

⇒ $z_0 = \sqrt[4]{8}\,\frac{1}{2}(1+\sqrt{3}i) = 2^{-\frac{1}{4}}(1+\sqrt{3}i)$ $z_1 = iz_0 = 2^{-\frac{1}{4}}(i-\sqrt{3})$

$z_2 = -z_0 = -2^{-\frac{1}{4}}(1+\sqrt{3}i)$ $z_3 = -z_1 = 2^{-\frac{1}{4}}(\sqrt{3}-i)$

b) $z^3 = 2-11i$ ⇒ $z = \sqrt[3]{2-11i}$ $|2-11i| = \sqrt{125} = 5\sqrt{5}$

$\phi = \arg(2-11i) = -\arctan\frac{11}{2}$ ⇒ $z_k = \sqrt[3]{2-11i} = \sqrt[6]{125}\,e^{-\frac{i}{3}\arctan\frac{11}{2}}\cdot\omega_{3,k}$

$z_0 = 2-i$ $z_1 = z_0\,\omega_{3,1} = (2-i)\frac{1}{2}(-1+\sqrt{3}i)$ $z_2 = z_0\,\omega_{3,2}$ ⇒

$z_2 = (2-i)\frac{1}{2}(-1-\sqrt{3}i)$

c) $z^2+(1+i)^i = 0$ ⇒ $z = \sqrt{-(1+i)^i}$ $|1+i| = \sqrt{2}$

$\arg(1+i) = \frac{\pi}{4}+2k\pi$ ⇒ $1+i = \sqrt{2}\,e^{i(\frac{\pi}{4}+2k\pi)} = e^{\frac{1}{2}\ln 2+i(\frac{\pi}{4}+2k\pi)}$

⇒ $(1+i)^i = e^{\frac{i}{2}\ln 2-\frac{\pi}{4}-k\pi}$ $k\in\mathbb{Z}$

$z = \sqrt{-(1+i)^i} = \pm i\,e^{\frac{i}{4}\ln 2}\,e^{-\frac{\pi}{8}-\frac{k}{2}\pi}$ $k\in\mathbb{Z}$